Properties and calculations of irreducible multiparty correlations

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Outline I

Motivation

- Quantum information makes multipartite quantum states becoming the research subject.
- The information viewpoint is essential in characterizing the correlation structure in a multipartite quantum state.

A typical problem solved by maximal entropy principle

Quantum state estimation

- How to uncover an unknown state of a qubit ?
- How much information obtained in the process ?

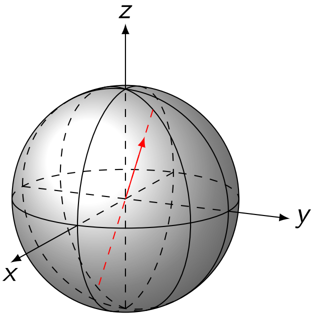
Quantum states of a qubit — Bloch sphere

- A quantum state of a qubit

$$\begin{aligned}\rho &= \frac{I+\vec{r}\cdot\vec{\sigma}}{2} \\ &= \frac{1+r}{2}|+_r\rangle\langle +_r| + \frac{1-r}{2}|-_r\rangle\langle -_r|\end{aligned}$$

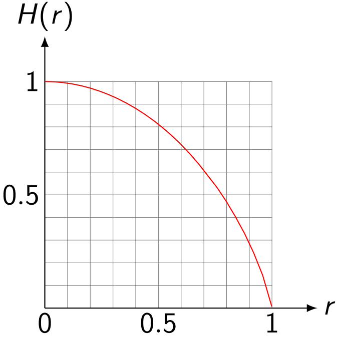
with $r \leq 1$ and $\langle \sigma_r \rangle = r$

- Geometric picture — Bloch sphere

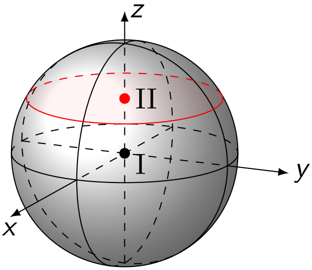


von Neumann entropy of a qubit's state

$$
\begin{aligned}
S(\rho) &= -\mathrm{Tr}(\rho \log_2 \rho) \\
&= -\frac{1+r}{2}\log_2\frac{1+r}{2} - \frac{1+r}{2}\log_2\frac{1+r}{2} \\
&= H(r)
\end{aligned}
$$



Equal probability distribution



1. An unknown state state of a qubit

$$\rho_{\mathrm{I}} = \frac{\int_{r\leq 1} \rho(\vec{r}) d\vec{r}}{\mathrm{Tr}(\int_{r\leq 1} \rho(\vec{r}) d\vec{r})} = \frac{I}{2}$$

2. Measurement of σ_z, obtain $\langle \sigma_z \rangle = z_0$

$$\rho_{\mathrm{II}} = \frac{\int_{r\leq 1, z=z_0} \rho(\vec{r}) d\vec{r}}{\mathrm{Tr}(\int_{r\leq 1, z=z_0} \rho(\vec{r}) d\vec{r})} = \frac{I + z_0 \sigma_z}{2}$$

Maximal entropy principle

1. An unknown state state of a qubit

$$\rho_{\mathrm{I}} = \arg\max_{r\leq 1} S(\rho(\vec{r}))$$

2. Measurement of σ_z, obtain $\langle\sigma_z\rangle = z_0$

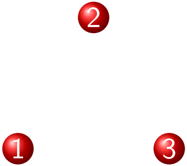
$$\rho_{\mathrm{II}} = \arg\max_{r\leq 1, z=z_0} S(\rho(\vec{r}))$$

3. The information obtained

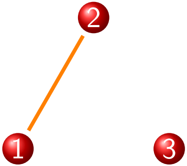
$$\Delta S(\mathrm{I}\rightarrow\mathrm{II}) = S(\rho_{\mathrm{I}}) - S(\rho_{\mathrm{II}})$$

Basic observations on correlation classification

- No correlation
  $\rho^{(123)} = \rho^{(1)} \otimes \rho^{(2)} \otimes \rho^{(3)}$

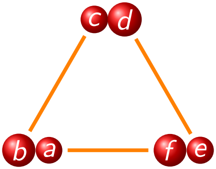


- No three-party correlation: $\rho^{(123)} = \rho^{(12)} \otimes \rho^{(3)}$



Reducible three-party correlation

Three partite quantum state $\rho^{(123)} = \rho^{(bc)} \otimes \rho^{(de)} \otimes \rho^{(fa)}$
with $1 = \{a, b\}, 2 = \{c, d\}, 3 = \{e, f\}$



Classification of measurements

1. One-party Hermitian operators
   - $\{O_i^{(1)} \otimes I^{(2)} \otimes I^{(3)}\} \rightarrow \rho^{(1)}$
   - $\{I^{(1)} \otimes O_j^{(2)} \otimes I^{(3)}\} \rightarrow \rho^{(2)}$
   - $\{I^{(1)} \otimes I^{(2)} \otimes O_k^{(3)}\} \rightarrow \rho^{(3)}$
2. Two-party Hermitian operators
   - $\{O_i^{(1)} \otimes O_j^{(2)} \otimes I^{(3)}\} \rightarrow \rho^{(12)}$
   - $\{I^{(1)} \otimes O_j^{(2)} \otimes O_k^{(3)}\} \rightarrow \rho^{(23)}$
   - $\{O_i^{(1)} \otimes I^{(2)} \otimes O_k^{(3)}\} \rightarrow \rho^{(13)}$
3. Three-party Hermitian operators
   - $\{O_i^{(1)} \otimes O_j^{(2)} \otimes O_k^{(3)}\} \rightarrow \rho^{(123)}$

Maximal Entropy Principle

1. Define

$$\begin{aligned}
A_0(\rho^{(123)}) &= \{\sigma^{(123)} | \sigma^{(123)} \in \mathcal{H}^{(123)}\} \\
A_1(\rho^{(123)}) &= \{\sigma^{(123)} | \sigma^{(1)} = \rho^{(1)}, \sigma^{(2)} = \rho^{(2)}, \sigma^{(3)} = \rho^{(3)}\} \\
A_2(\rho^{(123)}) &= \{\sigma^{(123)} | \sigma^{(12)} = \rho^{(12)}, \sigma^{(23)} = \rho^{(23)}, \sigma^{(13)} = \rho^{(13)}\} \\
A_3(\rho^{(123)}) &= \{\sigma^{(123)} | \sigma^{(123)} = \rho^{(123)}\}
\end{aligned}$$

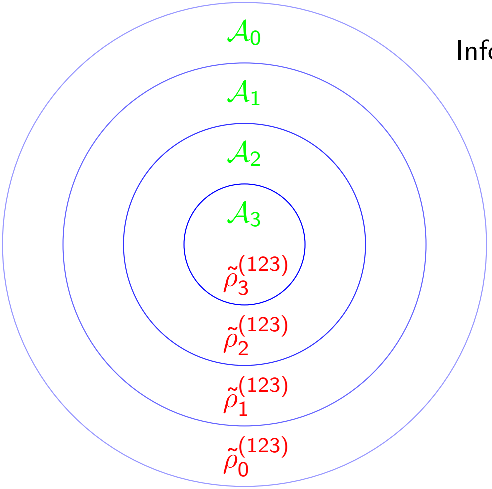
2. Find the state $\tilde{\rho}_m^{(123)}$ which satisfies

$$\tilde{\rho}_m^{(123)} = \arg\max_{\sigma^{(123)} \in A_m(\rho^{(123)})} S(\sigma^{(123)})$$

3. The irreducible m-party correlation is defined as

$$C_M^m(\rho^{(123)}) = S(\tilde{\rho}_m^{(123)}) - S(\tilde{\rho}_{m-1}^{(123)})$$

Demonstration of information hierarchy



Information via measurement
(Entropy decreasing)

Exponential form for states with maximal rank

Theorem

The exponential form of $\tilde{\rho}_m^{(123)}$ *for a state* $\rho^{(123)}$ *with maximal rank*

$$\tilde{\rho}_m^{(123)} = \exp\left(\sum_{N_0(\{ijk\})\geq 3-m} \theta_m^{ijk} O_i^{(1)} \otimes O_j^{(2)} \otimes O_k^{(3)}\right)$$

where $N_0(\{ijk\}) = \delta_{0i} + \delta_{0j} + \delta_{0k}$ *with* $O_0^{(n)} = \mathrm{I}^{(n)}$.

Remarks:

1. To judge whether a state with maximal rank has the irreducible three-party or two-party correlation
2. Still difficult to find out $\tilde{\rho}_m^{(123)}$ in general.
3. Need careful treatment with the state without maximal rank.

Corollary

1. $\tilde{\rho}_0^{(123)} = \frac{\mathrm{I}^{(1)}}{d_1} \frac{\mathrm{I}^{(2)}}{d_2} \frac{\mathrm{I}^{(3)}}{d_3}$
2. $\tilde{\rho}_1^{(123)} = \rho^{(1)} \otimes \rho^{(2)} \otimes \rho^{(3)}$
3. $\tilde{\rho}_3^{(123)} = \rho^{(123)}$

Definitions

- The degree of the total correlation

$$\begin{aligned} C_T(\rho^{(123)}) &\equiv S(\tilde{\rho}_1^{(123)}) - S(\tilde{\rho}_3^{(123)}) \\ &= S(\rho^{(1)}) + S(\rho^{(2)}) + S(\rho^{(3)}) - S(\rho^{(123)}). \end{aligned}$$

- The degree of local information

$$\begin{aligned} C_L(\rho^{(123)}) &\equiv S(\tilde{\rho}_0^{(123)}) - S(\tilde{\rho}_1^{(123)}) \\ &= \log_2(d_1 d_2 d_3) - S(\rho^{(1)}) - S(\rho^{(2)}) - S(\rho^{(3)}). \end{aligned}$$

Additivity of irreducible multiparty correlations

Theorem

$$C_M^m(\rho^{(123)} \otimes \sigma^{(456)}) = C_M^m(\rho^{(123)}) + C_M^m(\sigma^{(456)})$$

The uniqueness of $\tilde{\rho}_m^{(123)}$

Theorem

There exists a unique solution of $\tilde{\rho}_m^{(123)}$ for the state $\rho^{(123)}$ with maximal rank.

A piece of not so good news

A three-qubit state without irreducible three-party correlations $\xrightarrow{\text{Local operations}}$ A three-qubit state with irreducible three-party correlations

Remark: Local operations can transform two-party correlations into irreducible three-party correlations.

Correlation analysis: typical examples

1. Classically correlated state

$$\rho_C^{(123)} = \frac{1}{2}(|000\rangle\langle 000| + |111\rangle\langle 111|)$$

2. GHZ state

$$|\psi_G\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

3. Generalized GHZ states

$$|\psi_{GG}\rangle = \alpha|000\rangle + \beta|111\rangle$$

Explicit analysis of $\rho_C^{(123)}$

1. Another form of the state

$$\rho_C^{(123)} = \frac{1}{8}(I^{(1)}I^{(2)}I^{(3)} + Z^{(1)}Z^{(2)}I^{(3)} + I^{(1)}Z^{(2)}Z^{(3)} + Z^{(1)}I^{(2)}Z^{(3)})$$

2. The total correlation

$$C_T(\rho_C^{(123)}) = S(\rho_C^{(1)}) + S(\rho_C^{(2)}) + S(\rho_C^{(3)}) - S(\rho_C^{(123)}) = 2$$

Difficulties

The key point is to find $\tilde{\rho}_{C2}^{(123)}$

$$\tilde{\rho}_{C2}^{(123)} = \arg\max_{\{\lambda_{ijk}\}} S(\sigma(\{\lambda_{ijk}\}))$$

where

$$\begin{aligned}\sigma(\{\lambda_{ijk}\}) &= \rho_C^{(123)} + \frac{1}{8}(\lambda_{111}X^{(1)}X^{(2)}X^{(3)} \\ &\quad + \lambda_{112}X^{(1)}X^{(2)}X^{(3)} + \cdots + \lambda_{333}Z^{(1)}Z^{(2)}Z^{(3)})\end{aligned}$$

Difficulties:

1. Semi-positivity of $\sigma(\{\lambda_{ijk}\})$ is a complex condition.
2. $\rho_C^{(123)}$ without maximal rank, can not be written as an exponential form.

Basic idea of continuity approach

The multi-particle state without maximal rank can always be regarded as the limit of the multi-particle states with maximal rank.

Application to the state $\rho_C^{(123)}$

1. Construct a state with exponential form

$$\tilde{\rho}_{C2}^{(123)}(\lambda) = \exp\left(\eta + \lambda Z^{(1)} Z^{(2)} I^{(3)} + \lambda Z^{(1)} I^{(2)} Z^{(3)}\right)$$

2. Another form of the state

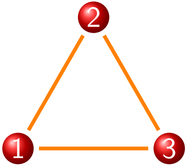
$$\begin{aligned}\tilde{\rho}_{C2}^{(123)}(\lambda) &= \frac{1}{8}\Big(1 + \tanh \lambda Z^{(1)} Z^{(2)} I^{(3)} + \tanh \lambda Z^{(1)} I^{(2)} Z^{(3)} \\ &\quad + \tanh^2 \lambda I^{(1)} Z^{(2)} Z^{(3)}\Big)\end{aligned}$$

3. The limit

$$\lim_{\lambda \to +\infty} \tilde{\rho}_{C2}^{(123)}(\lambda) = \tilde{\rho}_{C2}^{(123)}$$

The correlation structure in the state $\rho_C^{(123)}$

- $\rho_C^{(123)}$ does not have irreducible three-party correlation.



- The degree of the total correlation in $\rho_C^{(123)}$ is 2 bits.

The correlation structure in the state $\rho_G^{(123)}$

- The degree of the total correlation in $\rho_G^{(123)}$ is 3 bits.
- The degree of irreducible two-party correlation is 2 bits.
- The degree of irreducible three-party correlation is 1 bit.

Phase uncertainty and irreducible three-party correlation

Two-party measurement can not get the information about ϕ

$$|\psi_G(\phi)\rangle = \frac{1}{\sqrt{2}}(|000\rangle + e^{i\phi}|111\rangle)$$

The correlation structure in the state $\rho_{GG}^{(123)}$

- The degree of the total correlation in $\rho_{GG}^{(123)}$ is $3 \times H(|\alpha|^2)$ bits.
- The degree of irreducible two-party correlation is $2 \times H(|\alpha|^2)$ bits.
- The degree of irreducible three-party correlation is $H(|\alpha|^2)$ bit.

  The function

$$H(p) = -p \log_2 p - (1-p) \log_2 (1-p)$$

Our numerical algorithm on calculation of $\tilde{\rho}_2^{(123)}$ (I)

- Construct a series of state

$$\rho^{(123)}(p_0) = p_0 \frac{I^{(1)} I^{(2)} I^{(3)}}{8} + (1 - p_0)\rho^{(123)}$$

- Optimization parameters

$$\tilde{\rho}_2^{(123)}(p_0) = \frac{\exp(H_2(\{\lambda(p_0)\}))}{\mathrm{Tr}\exp(H_2(\{\lambda(p_0)\}))}$$

with

$$\begin{aligned} H_2(\{\lambda(p_0)\})) &= \lambda_{001}(p_0) I^{(1)} I^{(2)} X^{(3)} + \cdots + \lambda_{300}(p_0) Z^{(1)} I^{(2)} I^{(3)} \\ &\quad + \lambda_{011}(p_0) I^{(1)} X^{(2)} X^{(3)} + \cdots + \lambda_{330}(p_0) Z^{(1)} Z^{(2)} I^{(3)} \end{aligned}$$

Our numerical algorithm on calculation of $\tilde{\rho}_2^{(123)}$ (II)

- The conditions to determine $\lambda_{ijk}(p_0)$

$$
\begin{aligned}
\mathrm{Tr}(I^{(1)}I^{(2)}X^{(3)}\tilde{\rho}_2^{(123)}(p_0)) &= \mathrm{Tr}(I^{(1)}I^{(2)}X^{(3)}\rho_2^{(123)}(p_0)) \\
\vdots \qquad & \vdots \qquad \vdots \\
\mathrm{Tr}(Z^{(1)}I^{(2)}I^{(3)}\tilde{\rho}_2^{(123)}(p_0)) &= \mathrm{Tr}(Z^{(1)}I^{(2)}I^{(3)}\rho_2^{(123)}(p_0)) \\
\mathrm{Tr}(I^{(1)}X^{(2)}X^{(3)}\tilde{\rho}_2^{(123)}(p_0)) &= \mathrm{Tr}(I^{(1)}X^{(2)}X^{(3)}\rho_2^{(123)}(p_0)) \\
\vdots \qquad & \vdots \qquad \vdots \\
\mathrm{Tr}(Z^{(1)}Z^{(2)}I^{(3)}\tilde{\rho}_2^{(123)}(p_0)) &= \mathrm{Tr}(Z^{(1)}Z^{(2)}I^{(3)}\rho_2^{(123)}(p_0))
\end{aligned}
$$

- The initial parameters $\lambda_{ijk}(p_0)$ via continuity principle
  - $p_0(k) = 1 - \frac{k}{N} \quad k \in \{0, 1, \cdots, n\}$
  - initial $\lambda_{ijk}(k+1) =$ final $\lambda_{ijk}(k)$
  - $\lambda_{ijk}(0) = 0$

Three key elements in numerical algorithm

1. A state with non-maximal rank is regarded as the limit of a series of state with maximal rank
2. The exponential form is taken to ensure the positivity of a quantum state.
3. Continuity principle is used to give the initial trying parameters.

Numerical result (I)

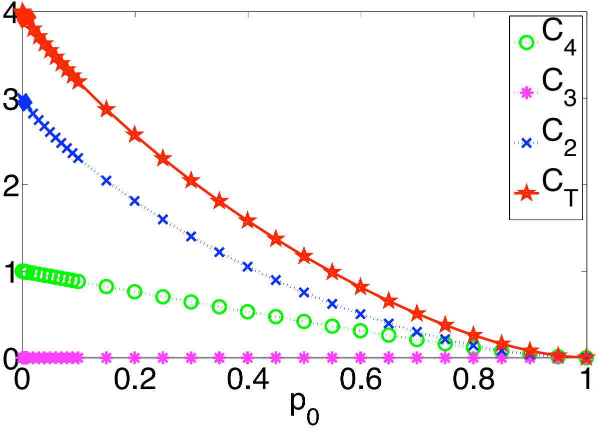


Figure: The degrees of irreducible multiparty correlations for the 4-qubit GHZ state $|\mathrm{GHZ}_4\rangle = \frac{1}{\sqrt{2}}(\prod_{i=1}^{4} \otimes |0\rangle_i + \prod_{i=1}^{4} \otimes |1\rangle_i)$.

Numerical result (II)

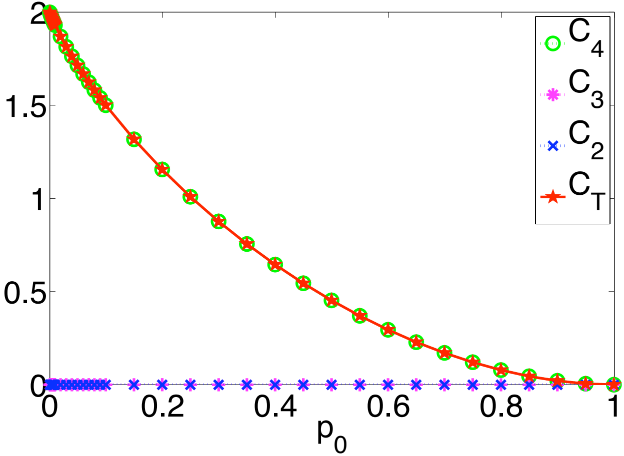


Figure: The degrees of irreducible multiparty correlations for the four-qubit Smolin state $\rho_{\rm sm}^{[4]} = \frac{1}{16}(I^{(1)}I^{(2)}I^{(3)}I^{(4)} + X^{(1)}X^{(2)}X^{(3)}X^{(4)} + Y^{(1)}Y^{(2)}Y^{(3)}Y^{(4)} + Z^{(1)}Z^{(2)}Z^{(3)}Z^{(4)})$.

Numerical result (III)

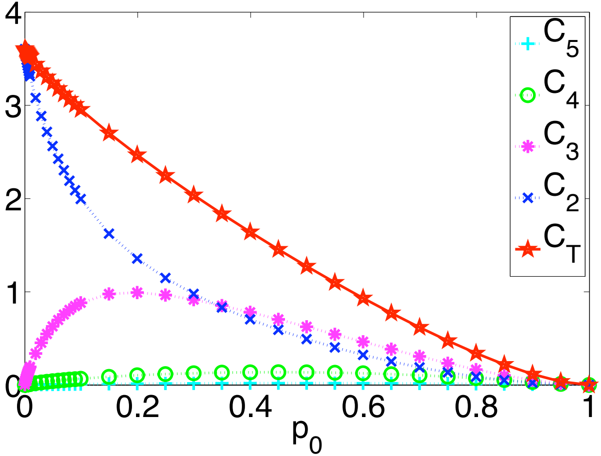


Figure: The degrees of irreducible multiparty correlations of the 5-qubit W state $|\mathrm{W}_5\rangle = \frac{1}{\sqrt{5}} \sum_{i=1}^{5} X^{(i)} \prod_{j=1}^{5} \otimes |0\rangle_j$.

Summary

1. The maximal entropy principle is regarded as a principle to estimate a state via partial information.
2. The formal definitions of the degree of irreducible k-particle correlations in an n-particle state are introduced, which gives a classification and quantification of multiparty correlations.
3. Four properties of irreducible multi-particle correlation are reviewed.
4. Adopting the continuity principle, we develop a method to apply a theorem to deal with the irreducible multi-particle correlations in the multi-particle states without maximal rank.
5. We successfully obtained the degrees of irreducible k-particle correlations in the n-qubit stabilizer states and the n-qubit generalized GHZ states.
6. We develop a numerical algorithm to calculate the degrees of irreducible multiparty correlations for quantum states up to five qubits.

Thanks for your attention!