第二届全国冷原子和量子信息青年学者学术讨论会

Fidelity susceptibility and quantum phase transitions

Speaker: Shi-Jian Gu

$$F(A,B)=\langle \Psi(A)|\Psi(B)\rangle$$

The Department of Physics and,
Institute of Theoretical Physics,
The Chinese University of Hong Kong,
Hong Kong, China

Zhengzhou, China, September

Collaborators:

Hai-Qing Lin (CUHK)

Wen-Long You (CUHK)

Ying-Wei Li (CUHK)

Ho-Man Kwok (CUHK)

Chun-Sing Ho(CUHK)

Wen-Long Lu(CUHK)

Shu Chen(IoP)

Xiaoguang Wang(ZJU)

Yuguang Chen(Tongji U.)

Chang-Qing Wu (Fudan U.)

Chang-Pu Sun (ITP)

Wen-Qiang Ning (Fudan U.)

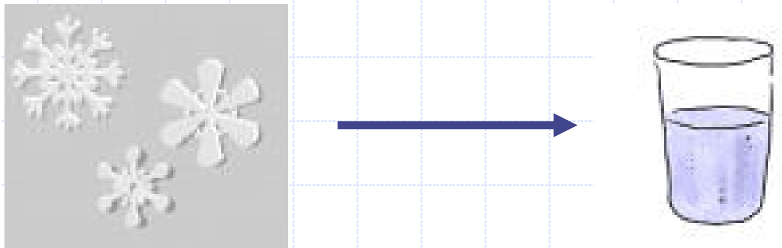
Shuo Yang (ITP)

A physical phenomenon can be understood from different point of view.

Content

Introduction: QPT

Thermal phase transitions: which is described by non-analytic behaviors of the thermal properties at the transition points, driven by thermal fluctuation.

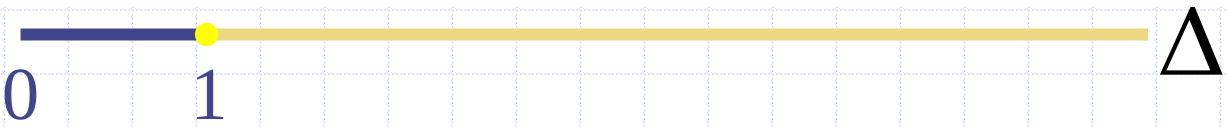


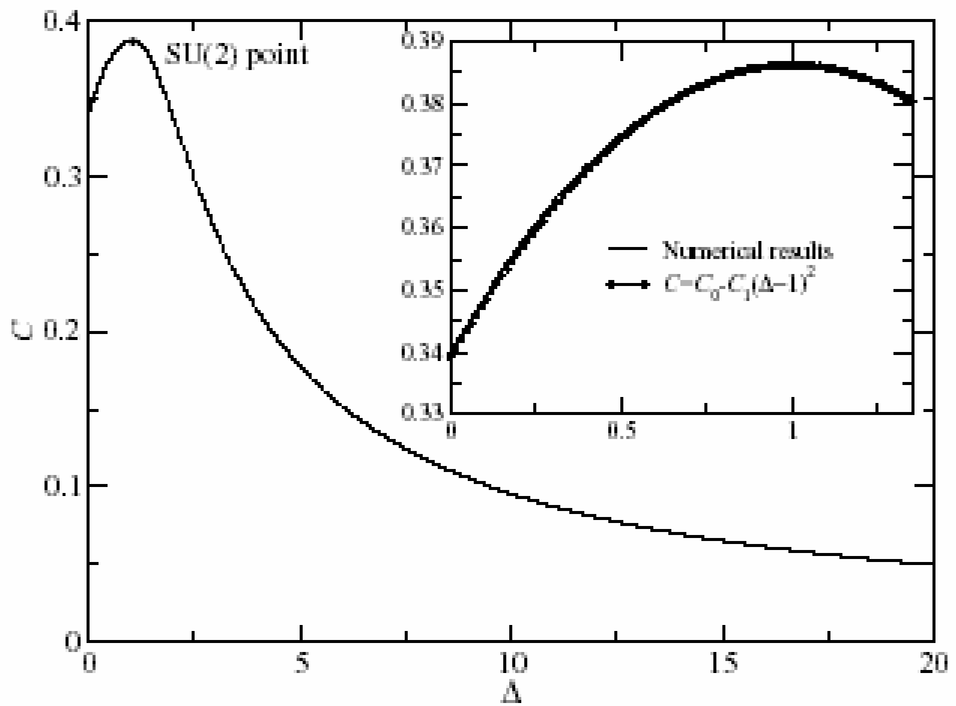
Quantum phase transitions: driven by the quantum fluctuations and are described by the non-analytic behaviors of the ground-state properties at the transition points.

- High Tc superconductor
- Mott-insulator transition in Hubbard model.

Introduction: QPT & quantum entanglement

$$\hat{H}_{\mathrm{XXZ}} = \sum_i \left(\hat{s}_i^x \hat{s}_{i+1}^x + \hat{s}_i^y \hat{s}_{i+1}^y + \Delta \hat{s}_i^z \hat{s}_{i+1}^z \right);$$





$$C = C_0 - C_1 (\Delta - 1)^2$$

$$C_0 = 2\ln 2 - 1$$

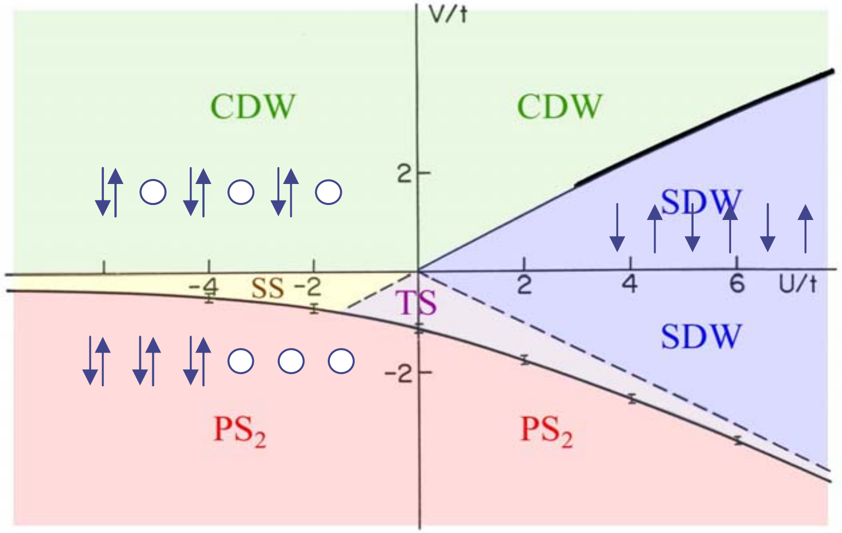
$$C_1 = 2\ln 2 - \frac{1}{2} - \frac{2}{\pi} - \frac{2}{\pi^2}$$

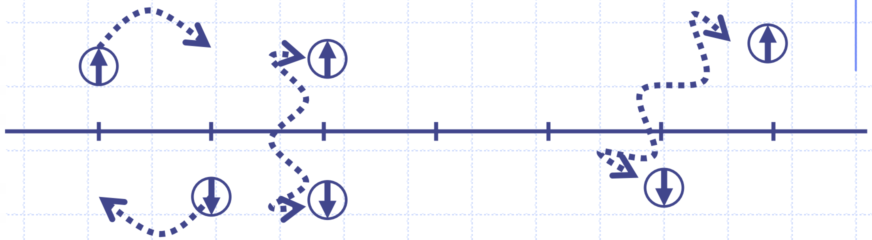
PRA, 68, 042330(2003).

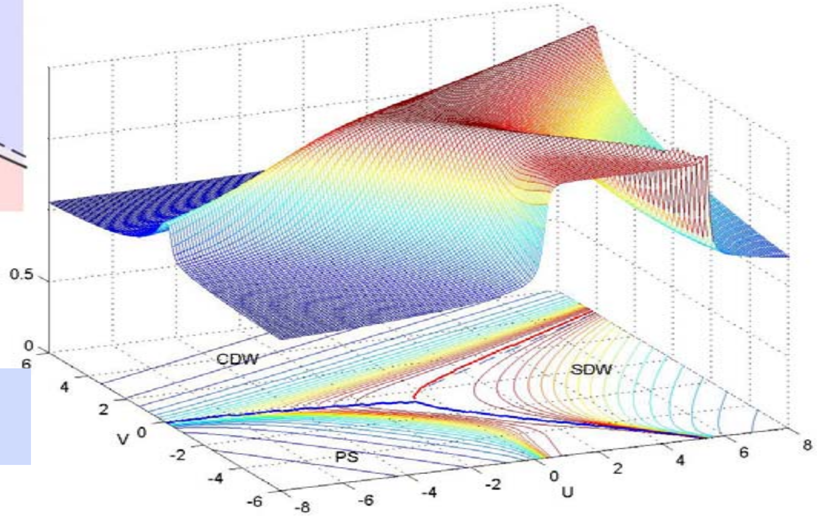
Introduction: QPT & quantum entanglement

The extended Hubbard model

$$H = -\sum_{\sigma,j,\delta} c^{+}_{j,\sigma} c_{j+\delta,\sigma} + U\sum_{j} n_{j,\uparrow} n_{j,\downarrow} + V\sum_{j} n_{j} n_{j+1}$$







Phys. Rev. Lett. 93, 086402 (2004).

Introduction: QPT & quantum entanglement

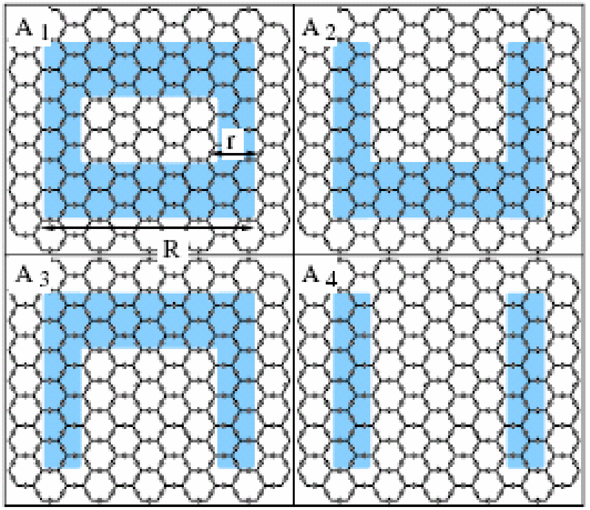
Detecting Topological Order in a Ground State Wave Function

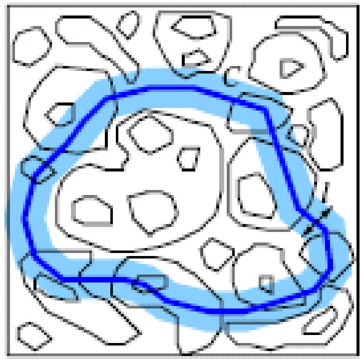
Michael Levin and Xiao-Gang Wen

Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

(Received 25 October 2005; published 24 March 2006)

A large class of topological orders can be understood and classified using the string-net condensation picture. These topological orders can be characterized by a set of data $(N, d_i, F^{ijk}_{lmn}, \delta_{ijk})$. We describe a way to detect this kind of topological order using only the ground state wave function. The method involves computing a quantity called the "topological entropy" which directly measures the total quantum dimension $D = \sum_i d_i^2$.





$$(S_1 - S_2) - (S_3 - S_4)$$

Introduction: QPT & Fidelity

H. T. Quan, Z. Song, X. F. Liu, P. Zanardi, and C. P. Sun, Phys. Rev. Lett. **96**, 140604 (2006).
P. Zanardi and N. Paunkovic, Phys. Rev. E **74**, 031123 (2006).
H. Q. Zhou and J. P. Barjaktarevic, arXiv: cond-mat/0701608 .
P. Zanardi, M. Cozzini, and P. Giorda, J. Stat. Mech. **2**, L02002 (2007).
M. Cozzini, P. Giorda, and P. Zanardi, Phys. Rev. B, **75**, 014439 (2007).
M. Cozzini, R. Ionicioiu, and P. Zanardi, Phys. Rev. B, **76**, 104420 (2007).
P. Buonsante and A. Vezzani, Phys. Rev. Lett. **98**, 110601 (2007).
P. Zanardi, H. T. Quan, X. Wang, and C. P. Sun, Phys. Rev. A **75**, 032109 (2007).
W. L. You, Y. W. Li, and S. J. Gu, Phys. Rev. E **76**, 022101 (2007).
P. Zanardi, P. Giorda, and M. Cozzini, Phys. Rev. Lett. **99**, 100603 (2007).
H. Q. Zhou, arXiv:0704.2945.
H. Q. Zhou, J. H. Zhao, and B. Li, arXiv:0704.2940;
L. C. Venuti and P. Zanardi, Phys. Rev. Lett. **99** , 095701 (2007).
S. Chen, L. Wang, S. J. Gu, and Y. Wang, Phys. Rev. E **76** 061108 (2007)
S. J. Gu, H. M. Kwok, W. Q. Ning, and H. Q. Lin, Phys. Rev. B **77**, 245109 (2008).
A. Tribedi and I. Bose, Phys. Rev. A **77**, 032307 (2008).
P. Zanardi, L. C. Venuti, and P. Giorda , Phys. Rev. A 76, 062318 (2007).
M. F. Yang, Phys. Rev. B **76**, 180403 (R) (2007).
Y. C. Tzeng and M. F. Yang, Phys. Rev. A **77**, 012311 (2008).
W. Q. Ning, S. J. Gu, Y. G. Chen, C. Q. Wu, and H. Q. Lin, J. Phys.: Condens. Matter **20**, 235236 (2008).
N. Paunković, P. D. Sacramento, P. Nogueira, V. R. Vieira, and V. K. Dugaev, Phys. Rev. A **77**, 052302 (2008).
N. Paunković and V. R. Vieira, Phys. Rev. E **77**, 011129 (2008).
H. M. Kwok, W. Q. Ning, S. J. Gu, and H. Q. Lin, arXiv:0710.2581.
H. Q. Zhou, R. Orus, and G. Vidal, Phy. Rev. Lett. **100** 080601 (2008).
H. Q. Zhou, J. H. Zhao, H. L. Wang, and B. Li, arXiv: 0711.4651.
J. O. Fjærestad, arXiv:0712.3439.
S. Chen, L. Wang, Y. Hao, and Y. Wang, Phys. Rev. A **77**, 032111 (2008).
D. F. Abasto, N. T. Jacobson, and P. Zanardi, Phys. Rev. A **77**, 022327 (2008).
L. C. Venuti, M. Cozzini, P. Buonsante, F. Massel, N. Bray-Ali, and P. Zanardi, arXiv:0801.2473.

Introduction: QPT & Fidelity

X. Wang, Z. Sun, and Z. D. Wang, arXiv:0803.2940
A. Hamma, W. Zhang, S. Haas, and D. A. Lidar, Phys. Rev. B **77**, 155111 (2008).
J. H. Zhao and H. Q. Zhou, arXiv:0803.0814.
S. Yang, S. J. Gu, C. P. Sun, and H. Q. Lin, Phys. Rev. A **78**, 012304 (2008).
D. F. Abasto, A. Hamma, and P. Zanardi, Phys. Rev. A **78**, 010301(R) (2008).
T. C. Tzeng, H. H. Hung, Y. C. Chen, and M. F. Yang, Phys. Rev. A **77**, 062321 (2008).
J. Zhang, X. Peng, N. Rajendran, and D. Suter, Phys. Rev. Lett. **100**, 100501 (2008).
H. M. Kwok, C. S. Ho, and S. J. Gu, arXiv:0805.3885.
J. Ma, L. Xu, H. Xiong, and X. Wang, arXiv:0805.4062.
H. M. Kwok, *Quantum fidelity and fidelity in many-body systems*, MPhil Thesis.
H. T. Quan and F. M. Cucchietti, arXiv:0806.4633.
L. C. Venuti, H. Saleur, P. Zanardi, arXiv:0807.0104.
X. M. Lu, Z. Sun, X. Wang, P. Zanardi, arXiv:0807.1370.
S. J. Gu and H. Q. Lin, arXiv:0807.3491.
Z. Ma, F. L. Zhangg, and J. L. Chen, arXiv:0808.0984.
J. Zhang, F. M. Cucchietti, C. M. Chandrashekar, M. Laforest, C. A. Ryan, M. Ditty, A. Hubbard, J. K. Gamble, and R. Laflamme, arXiv:0808.1536.
J. Ma, L. Xu, and X. Wang, arXiv:0808.1816.
H. N. Xiong, J. Ma, Z. Sun, and X. Wang, arXiv:0808.1817.

Introduction: Loschmidt echo in many-body system

F. M. Cucchietti, S. Fernandez-Vidal, and J. Pablo Paz, Phys. Rev. A **75**, 032337 (2007),

Y. C. Ou and H. Fan, J. Phys. A: Math. Theor. **40**, 2455(2007).

Z. G. Yuan, P. Zhang, and S. S. Li, Phys. Rev. A **75**, 012102 (2007).

D. Rossini, T. Calarco, V. Giovannetti, S. Montangero, and R. Fazio, Phys. Rev. A **75**, 032333 (2007).

H. T. Quan, Z. D. Wang, and C. P. Sun, Phys. Rev. A **76**, 012104 (2007).

L. C. Wang, X. L. Huang, X. X. Yi, Phys. Lett. A **368**, 362 (2007).

Y. C. Li and S. S. Li, Phys. Rev. A **76**, 032117 (2007).

X. X. Yi, H. Wang, and W. Wang, Euro. Phys. J. D **45**, 355 (2007).

Z. G. Yuan, P. Zhang, and S. S. Li, Phys. Rev. A **76**, 042118 (2007).

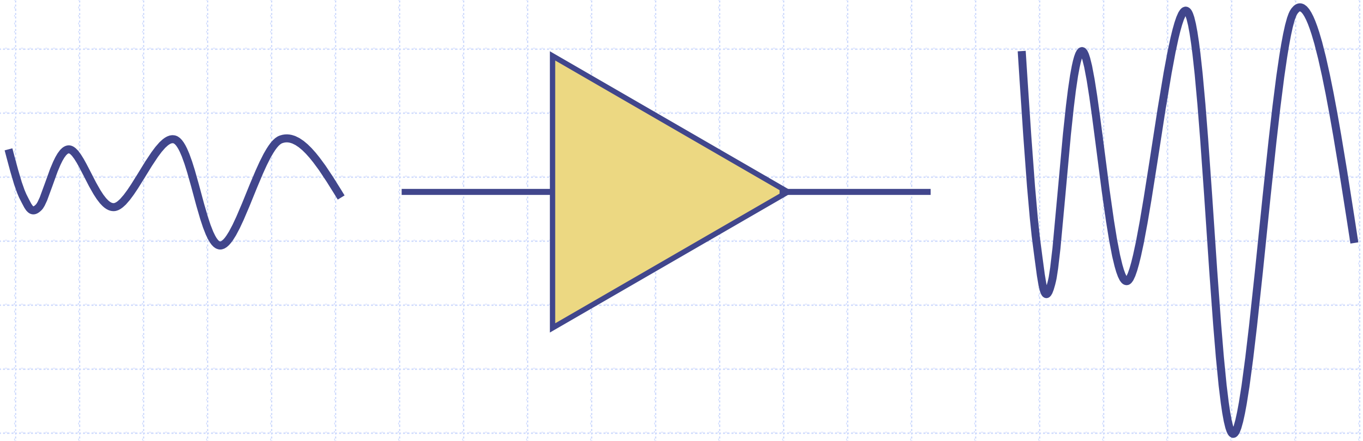
C. Cormick and J. Pablo Paz, Phys. Rev. A **77**, 022317 (2008).

D. Rossini, P. Facchi, R. Fazio, G. Florio, D. A. Lidar, S. Pascazio, F. Plastina, and P. Zanardi, Phys. Rev. A **77**, 052112 (2008).

W. G. Wang, J. Liu, and B. Li, Phys. Rev. E **77**, 056218(2008).

C. Y. Lai, J. T. Hung, C. Y. Mou, and P. Chen, Phys. Rev. B **77**, 205419 (2008).

Introduction: classical fidelity



Definition

$$\rho = p_1|1\rangle\langle 1| + p_2|2\rangle\langle 2| + \cdots + p_N|N\rangle\langle N|$$

$$\sigma = q_1|1\rangle\langle 1| + q_2|2\rangle\langle 2| + \cdots + q_N|N\rangle\langle N|$$

$$F = \sum_i \sqrt{p_i q_i}$$

Introduction: quantum fidelity

A. Uhlmann, Rep. Math. Phys. 9, 273 (1976)

$$F(\Psi',\Psi) = |\langle\Psi'|\Psi\rangle| \qquad \mathbf{a}\cdot\mathbf{b} = ab\cos(\theta)$$

Axioms

R. Jozsa, J. Mod. Opt. **41**, 2315 (1994).

$$\begin{aligned}
0 &\le F(\Psi',\Psi) \le 1,\\
F(\Psi',\Psi) &= F(\Psi,\Psi'),\\
F(U\Psi',U\Psi) &= F(\Psi',\Psi),\\
F(\Psi_1\otimes\Psi_2,\Psi_1'\otimes\Psi_2') &= F(\Psi_1',\Psi_1)F(\Psi_2',\Psi_2),
\end{aligned}$$

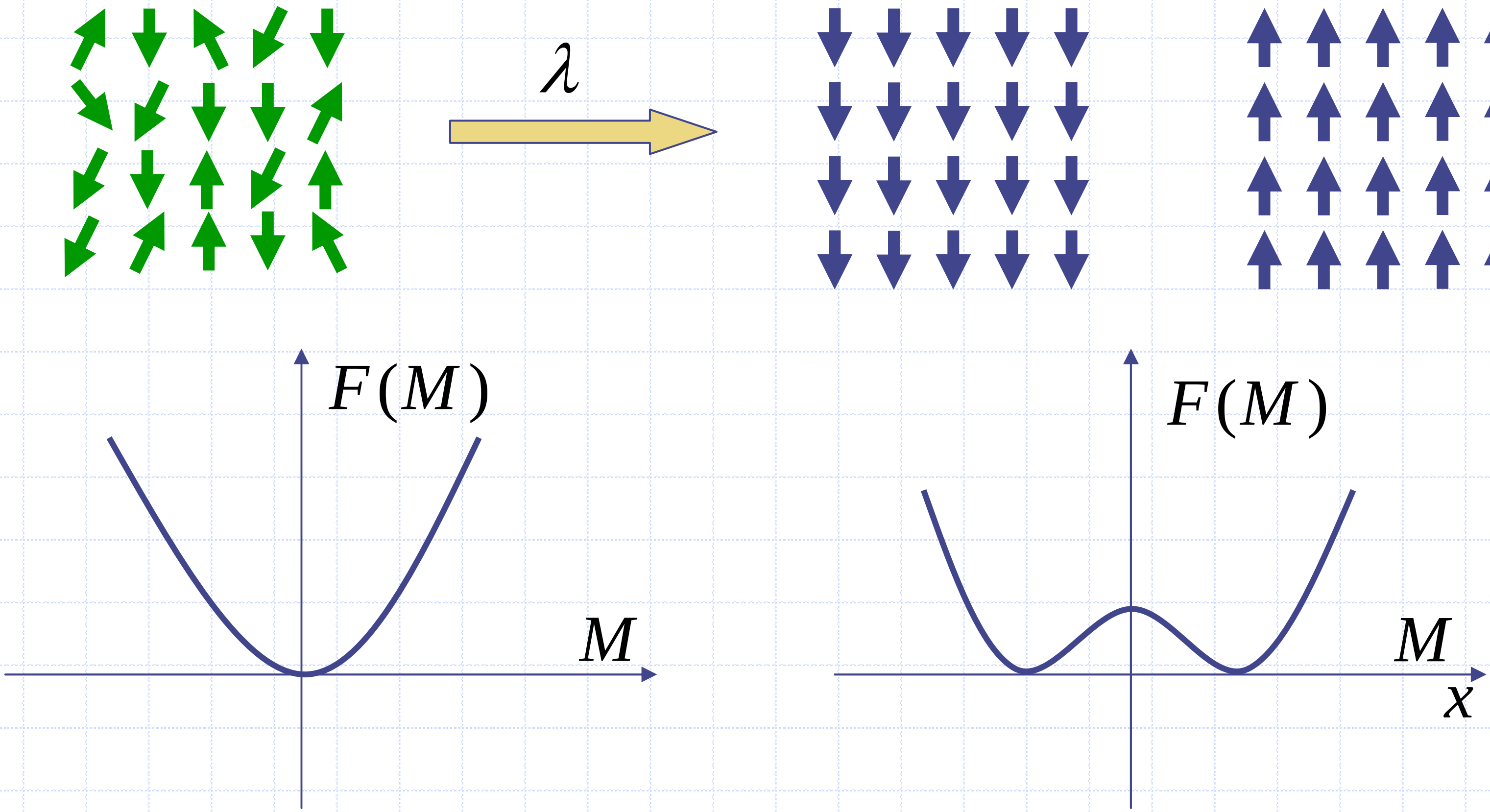
Example:

$$\begin{aligned}
|\Psi(\theta)\rangle &= \cos\theta|\uparrow\downarrow\rangle + \sin\theta|\downarrow\uparrow\rangle,\\
|\Psi(\theta')\rangle &= \cos\theta'|\uparrow\downarrow\rangle + \sin\theta'|\downarrow\uparrow\rangle,
\end{aligned}$$

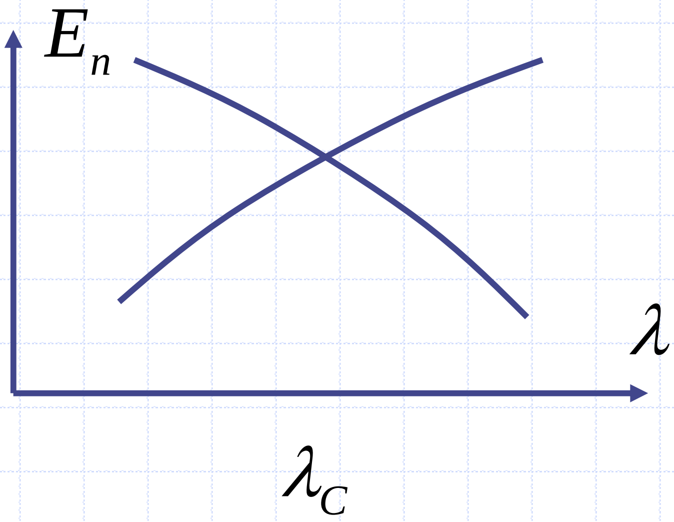
$$F(\Psi(\theta'),\Psi(\theta)) = |\cos(\theta-\theta')|.$$

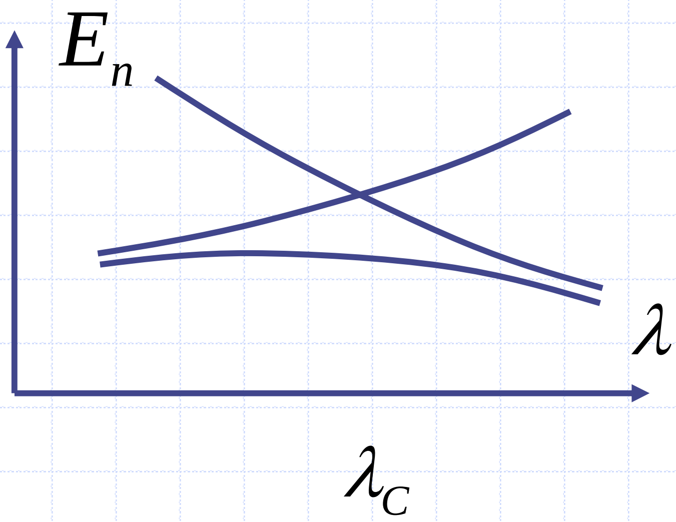
Introduction: traditional method

Landau's symmetry-breaking theory

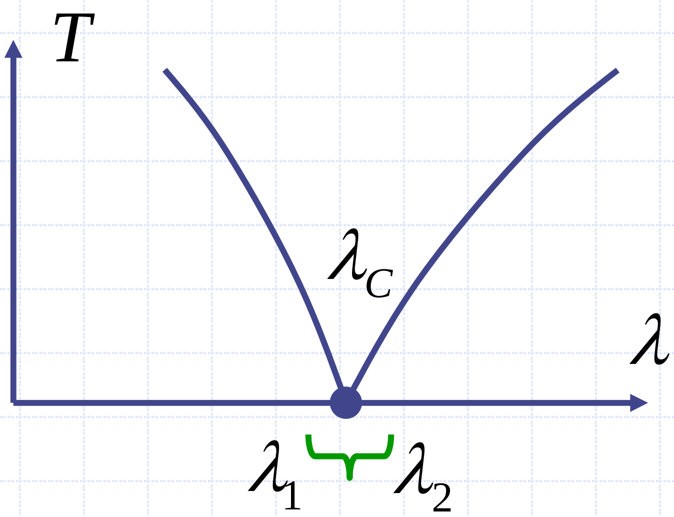


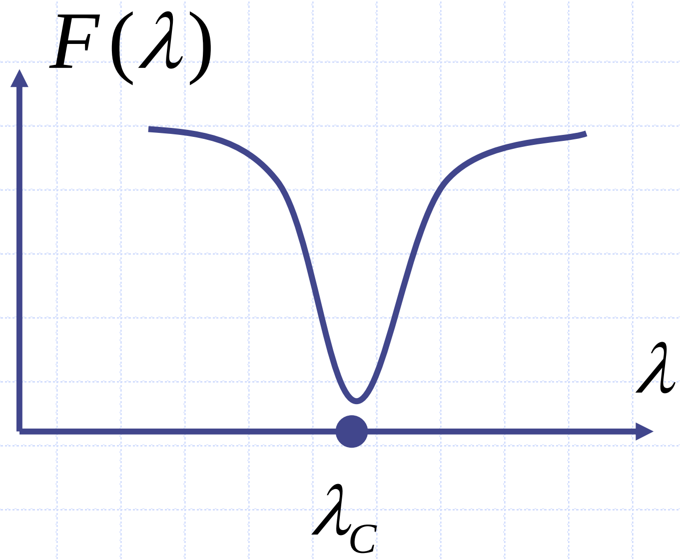
Introduction: information perspective





Introduction: information perspective





$$\psi(\lambda_1) \qquad \psi(\lambda_2)$$

$$F = |\langle \psi(\lambda_1) | \psi(\lambda_2) \rangle|$$

$$F(\rho, \sigma) = \sqrt{\rho^{1/2} \sigma \rho^{1/2}}$$

$$\rho = \begin{pmatrix} a & \\ & b \end{pmatrix} \qquad \sigma = \begin{pmatrix} c & \\ & d \end{pmatrix}$$

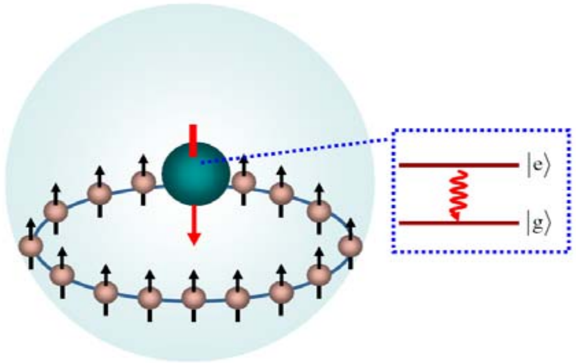
$$F(\rho, \sigma) = \sqrt{ac} + \sqrt{bd}$$

Introduction: QPT & Fidelity

Ising model

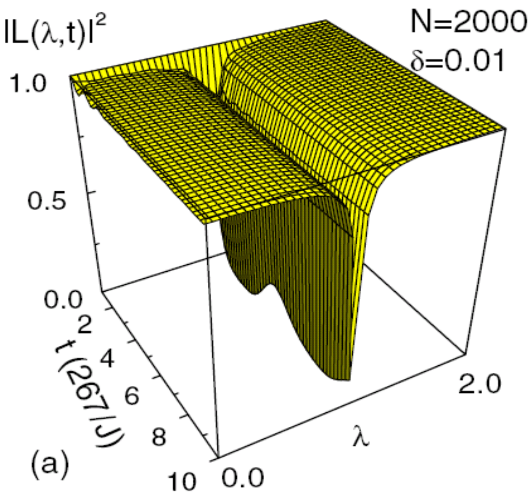
PRL **96,** 140604 (2006)

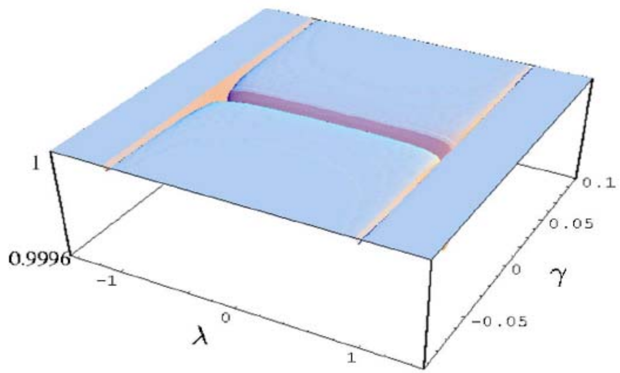
$$H(\lambda, \delta) = -J\sum_j (\sigma_j^z \sigma_{j+1}^z + \lambda\sigma_j^x + \delta|e\rangle\langle e|\sigma_j^x),$$



$$L(\lambda, t) = |\langle\varphi_g(t)|\varphi_e(t)\rangle|^2.$$

$$\hat{H}(\gamma,\lambda) = -\sum_{i=-M}^{M}\left(\frac{1+\gamma}{2}\hat{\sigma}_i^x\hat{\sigma}_{i+1}^x + \frac{1-\gamma}{2}\hat{\sigma}_i^y\hat{\sigma}_{i+1}^y + \frac{\lambda}{2}\hat{\sigma}_i^z\right).$$





PHYSICAL REVIEW E 74, 031123 (2006)

Relevant works:

1. W. L. You, Y. W. Li, and S. J. Gu, Phys. Rev. E **76**, 022101 (2007).
2. S. Chen, L. Wang, S. J. Gu, and Y. P. Wang, Phys. Rev. E **76**, 061108 (2007).
3. S. J. Gu, H. M. Kwok, W. Q. Ning, and H. Q. Lin, Phys. Rev. B 77, 245109 (2008)
4. H. M. Kwok, W. Q. Ning, S. J. Gu, and H. Q. Lin, PRE accepted, arXiv: 0710.2581.
5. W. Q. Ning, S. J. Gu, C. Q. Wu, and H. Q. Lin, J. Phys.: Condens. Matter 20 235236 (2008).
6. S. Yang, S. J. Gu, C. P. Sun, and H. Q. Lin, Phys. Rev. A 78, 012304 (2008)

Relevant works:

1. Partial-state fidelity and quantum phase transitions induced by continuous level crossing
   Ho-Man Kwok, Chun-Sing Ho, and Shi-Jian Gu
2. Dimension of fidelity susceptibility in quantum phases
3. Scaling of reduced fidelity susceptibility in the one-dimensional transverse-field XY model
   Wen-Long You,Wen-Long Lu, Xiaoguang Wang, and Shi-Jian Gu
4. Density-functional fidelity approach to quantum phase transitions,
   Shi-Jian Gu, preprint.

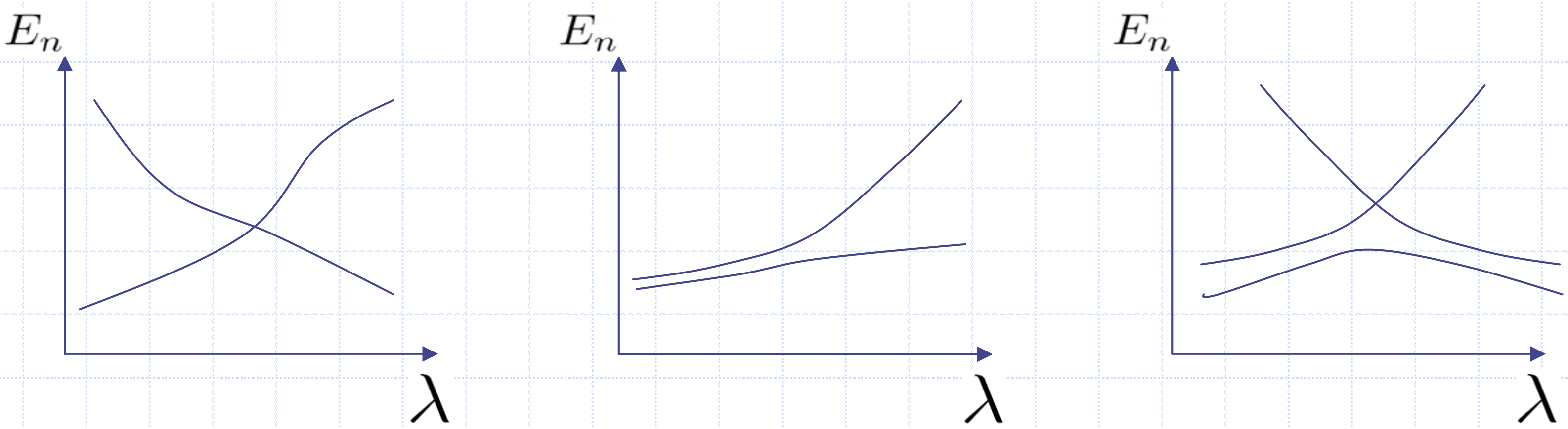
Content

Occurrence of the quantum phase transitions

How does QPT happen for a general quantum system

$$H(\lambda) = H_0 + \lambda H_I,$$

$$H(\lambda)|\Psi_n(\lambda)\rangle \ = \ E_n|\Psi_n(\lambda)\rangle$$



Perturbation method in quantum mechanics

Fidelity susceptibility

$$|\Psi_0(\lambda+\delta\lambda)\rangle = |\Psi_0(\lambda)\rangle + \delta\lambda \sum_{n\neq 0} \frac{H_{n0}(\lambda)|\Psi_n(\lambda)\rangle}{E_0(\lambda)-E_n(\lambda)}$$

$$H_{n0} = \langle\Psi_n(\lambda)|H_I|\Psi_0(\lambda)\rangle.$$

$$F_i(\lambda,\delta) = |\langle\Psi_0(\lambda)|\Psi_0(\lambda+\delta)\rangle|$$

$$\frac{1}{F_i^2} = 1 + \delta\lambda^2 \sum_{n\neq 0} \frac{|\langle\Psi_n(\lambda)|H_I|\Psi_0(\lambda)\rangle|^2}{[E_n(\lambda)-E_0(\lambda)]^2}$$

$$\chi_F \equiv \lim_{\delta\lambda\to 0} \frac{-2\ln F_i}{\delta\lambda^2} \qquad \chi_F(\lambda) = \sum_{n\neq 0} \frac{|\langle\Psi_n(\lambda)|H_I|\Psi_0(\lambda)\rangle|^2}{[E_n(\lambda)-E_0(\lambda)]^2}$$

Perturbation method in quantum mechanics

Fidelity susceptibility: what is the physics

$$\chi_F(\omega) = \sum_{n\neq 0} \frac{|\langle\Psi_n|H_I|\Psi_0\rangle|^2}{[E_n - E_0]^2 + \omega^2}$$

$$\frac{\partial \chi_F(\tau)}{\partial \tau} = -\pi\left[\langle\Psi_0|H_I(\tau)H_I(0)|\Psi_0\rangle - \langle\Psi_0|H_I|\Psi_0\rangle^2\right]\theta(\tau)$$

$$+\,\pi\left[\langle\Psi_0|H_I(0)H_I(\tau)|\Psi_0\rangle - \langle\Psi_0|H_I|\Psi_0\rangle^2\right]\theta(-\tau)$$

Fidelity susceptibility = dynamic structure factor

Fidelity and dynamics structure factor

Fidelity susceptibility: how to compute

$$\chi_F = \int \tau \left[\langle \Psi_0 | H_I(\tau) H_I(0) | \Psi_0 \rangle - \langle \Psi_0 | H_I | \Psi_0 \rangle^2 \right] d\tau$$

$$\langle \Psi_0 | H_I(\tau) H_I(0) | \Psi_0 \rangle = \sum_n \frac{\tau^n (-1)^n}{n!} e^{\tau E_0} \langle \Psi_0 | H_I H^n H_I | \Psi_0 \rangle .$$

To higher order

$$\chi_F = \frac{1}{H_I^{00}} \sum_{i,j>0} \frac{H_I^{0i} H_I^{ij} H_I^{j0}}{(E_i - E_0)(E_j - E_0)} - \frac{E_0^{(3)}}{E_0^{(1)}} .$$

S. Chen, L. Wang, Y. Hao, and Y. Wang, Phys. Rev. A **77**, 032111 (2008).

L. C. Venuti, M. Cozzini, P. Buonsante, F. Massel, N. Bray-Ali, and P. Zanardi, arXiv:0801.2473.

Extension to thermal phase transitions

Fidelity susceptibility: extension to TPT

P. Zanardi, H. T. Quan, X. Wang, and C. P. Sun, Phys. Rev. A **75**, 032109 (2007).

$$F_i(\beta,\delta) = \frac{Z(\beta)}{\sqrt{Z(\beta-\delta\beta/2)Z(\beta+\delta\beta/2)}},$$

$$Z(\beta) = \sum_n e^{-\beta E_n} = \sum_E g(E)e^{-\beta E}.$$

You, Li, and Gu, PRE, 76, 022101 (2007)

$$\chi_F = \left.\frac{-2\ln F_i}{\delta\beta^2}\right|_{\delta\beta\to 0} = \frac{C_v}{4\beta^2} \qquad C_v = \beta^2(\langle E^2\rangle - \langle E\rangle^2)$$

$$\chi_F = \left.\frac{-2\ln F_i}{\delta h^2}\right|_{\delta h\to 0} = \frac{\beta\chi}{4} \qquad \chi = \beta(\langle \dot{M}^2\rangle - \langle \dot{M}\rangle^2)$$

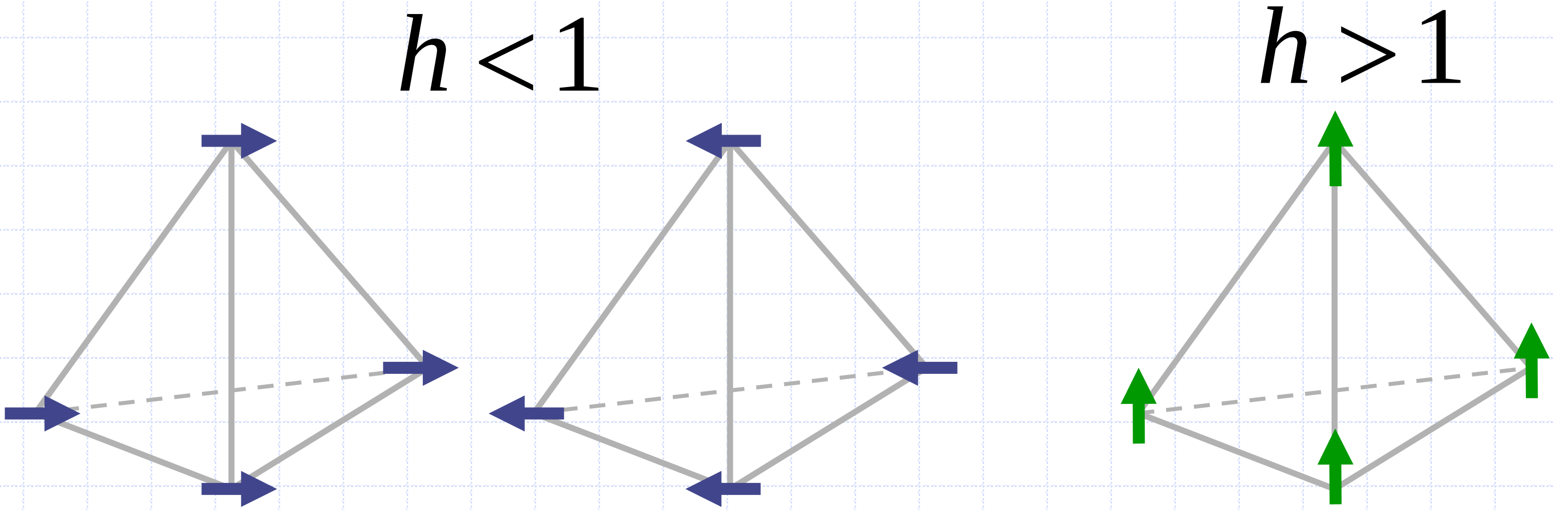
Content

Application: the Lipkin-Meshkov-Glick model

Hamiltonian

$$H = -\frac{\lambda}{N}\sum_{i<j}\left(\sigma_x^i\sigma_x^j + \gamma\sigma_y^i\sigma_y^j\right) - h\sum_i \sigma_z^i,$$

Ground phases (ferromagnetic)



Application: the LMG model

If $h > 1$

$$\begin{aligned} S_z &= S - a^\dagger a, \\ S_+ &= (2S - a^\dagger a)^{1/2} a \end{aligned}$$

The Hamiltonian in terms of bosons

$$H = -hN + [2(h-1) + \eta] a^\dagger a - \frac{\eta}{2}\left(a^{\dagger 2} + a^2\right)$$

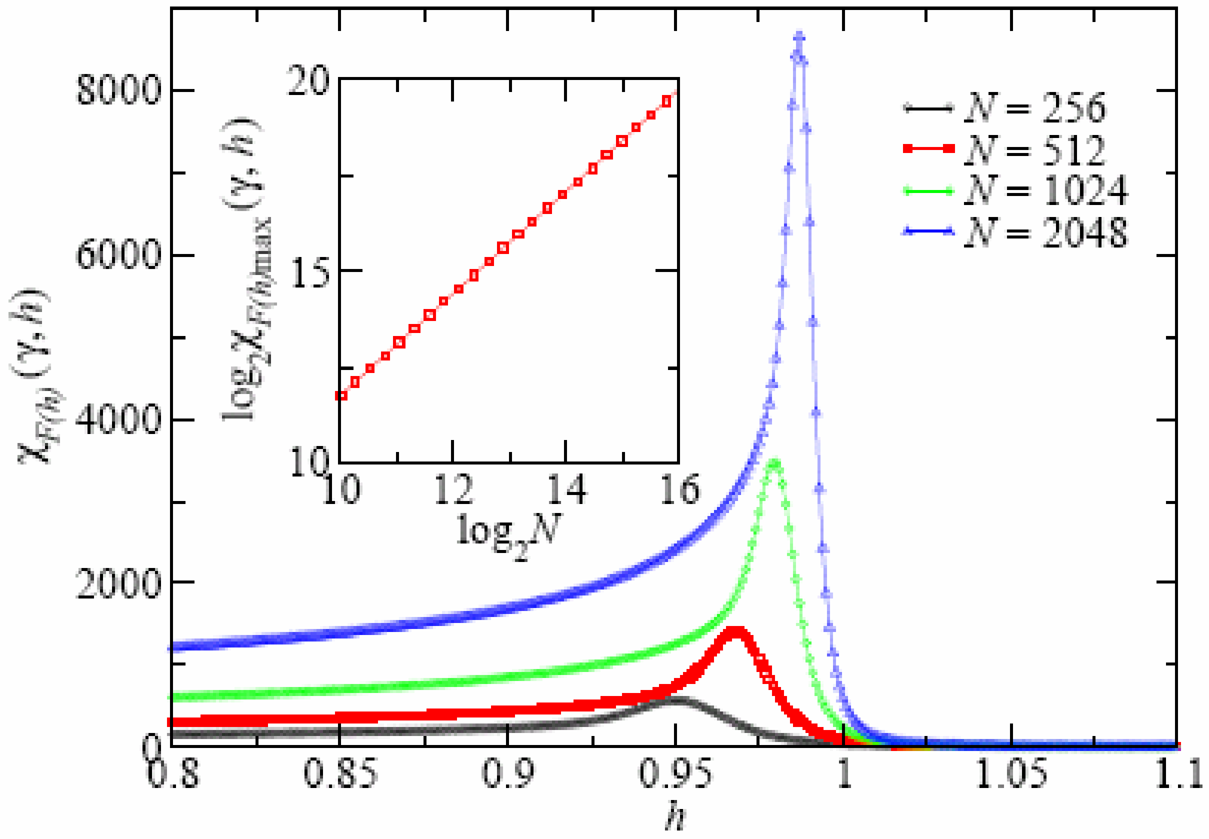
$$\begin{aligned} a^\dagger &= \cosh(\Theta/2) b^\dagger + \sinh(\Theta/2) b, \\ a &= \sinh(\Theta/2) b^\dagger + \cosh(\Theta/2) b, \end{aligned}$$

The diagonalized form

$$H = -h(N+1) + 2\sqrt{(h-1)(h-1+\eta)}\left(b^\dagger b + \frac{1}{2}\right)$$

Application: the LMG model

$$\gamma = 0.5$$



Universality class described by the FS

The fidelity susceptibility

L. C. Venuti and P. Zanardi, Phys. Rev. Lett. **99**, 095701 (2007).

$$\chi_F = \int \tau \left[\langle\Psi_0|H_I(\tau)H_I(0)|\Psi_0\rangle - \langle\Psi_0|H_I|\Psi_0\rangle^2\right] d\tau$$

$$r' = s\,r,\quad \tau' = s^{\zeta}\tau,\quad V(r') = s^{-\Delta_V}V(r)$$

$$\chi_F \sim L^{2\Delta_V - 2\zeta}$$

S. J. Gu, H. M. Kwok, W. Q. Ning, and H. Q. Lin, Phys. Rev. B **77**, 245109 (2008).

$$\frac{\chi_{F(\lambda)}(\lambda)}{N} \propto \frac{1}{|\lambda_c - \lambda|^{\alpha}}$$

$$N = L^d$$

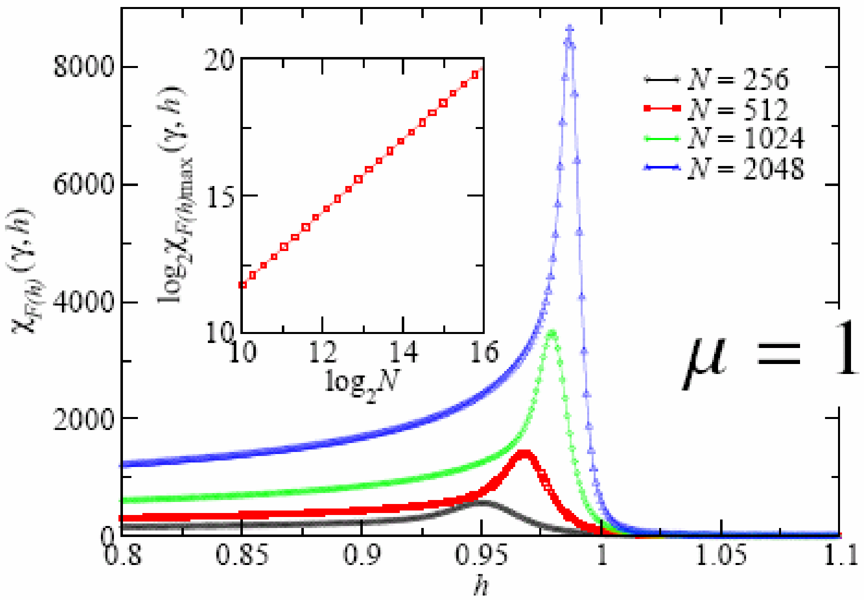
Application: asymmetric Hubbard model

Scaling ansatz: d (adiabatic dimension)

$$\frac{\chi(\lambda, L)}{L^{d^{\pm}}} = \frac{A}{L^{-\mu+d^{\pm}} + B(\lambda - \lambda_{\max})^{\alpha^{\pm}}}$$

$$\chi(\lambda = \lambda_{\max}) \propto L^{\mu}.$$

$$\frac{\chi_F}{L^{d_a^{\pm}}} \sim \frac{1}{|\lambda - \lambda_c|^{\alpha^{\pm}}}$$



$\mu = 1.33$

Application: asymmetric Hubbard model

Universal function

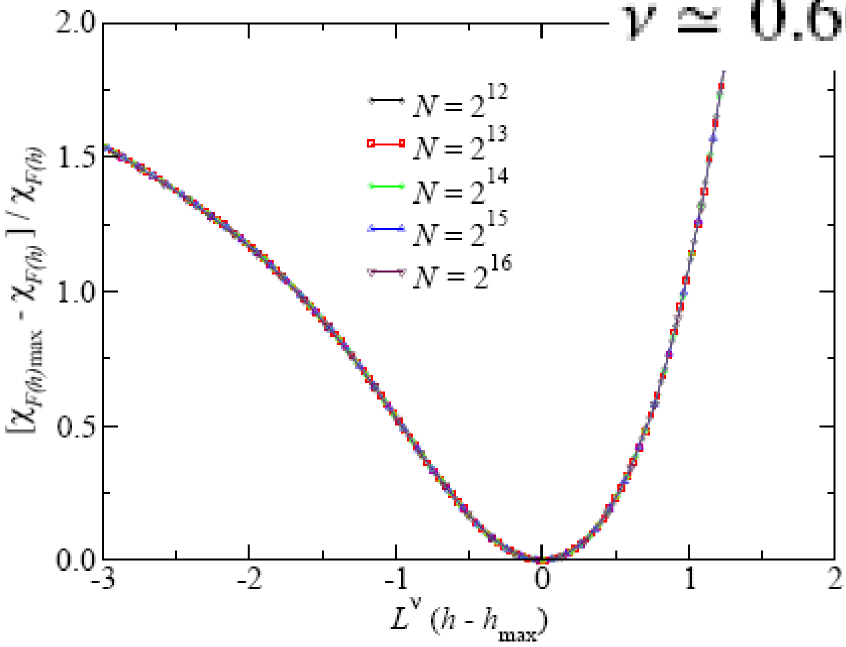
$$\frac{\chi_{F(\lambda)}(\lambda=\lambda_{\max},L)-\chi_{F(\lambda)}(\lambda,L)}{\chi_{F(\lambda)}(\lambda,L)}=f[L^{\nu}(\lambda-\lambda_{\max})]$$

$$\nu \simeq 0.665$$

Critical exponent

$$\alpha^{\pm}=\frac{\mu-d^{\pm}}{\nu}$$

$$\alpha=\begin{cases}2, h>1\\ \frac{1}{2}, 0\le h<1\end{cases}$$



Application: the LMG model

If $h > 1$

$$-\sum_i \sigma_z^i = -2S_z$$

$$\chi_{F(h)}(\eta, h > 1) = \frac{\eta^2}{32(h-1)^2(h-1+\eta)^2}$$

If $h < 1$

$$\frac{\chi_{F(h)}(\eta, h < 1)}{N} = \frac{1}{4\sqrt{(1-h^2)\eta}}$$

Exponents

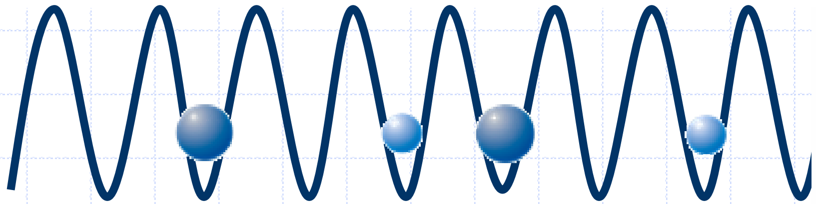
$$\alpha = \begin{cases} 2, h > 1 \\ \frac{1}{2}, 0 \le h < 1 \end{cases}$$

Application: asymmetric Hubbard model

$$H = -\sum_{j=1}^{L} \sum_{\delta=\pm 1} \sum_{\sigma} t_\sigma c^\dagger_{j,\sigma} c_{j+\delta,\sigma} + U \sum_{j=1}^{L} n_{j,\alpha} n_{j,\beta}.$$

$\sigma = \alpha, \beta$

$t_\alpha = 1, \ t_\beta \in [0, 1]$

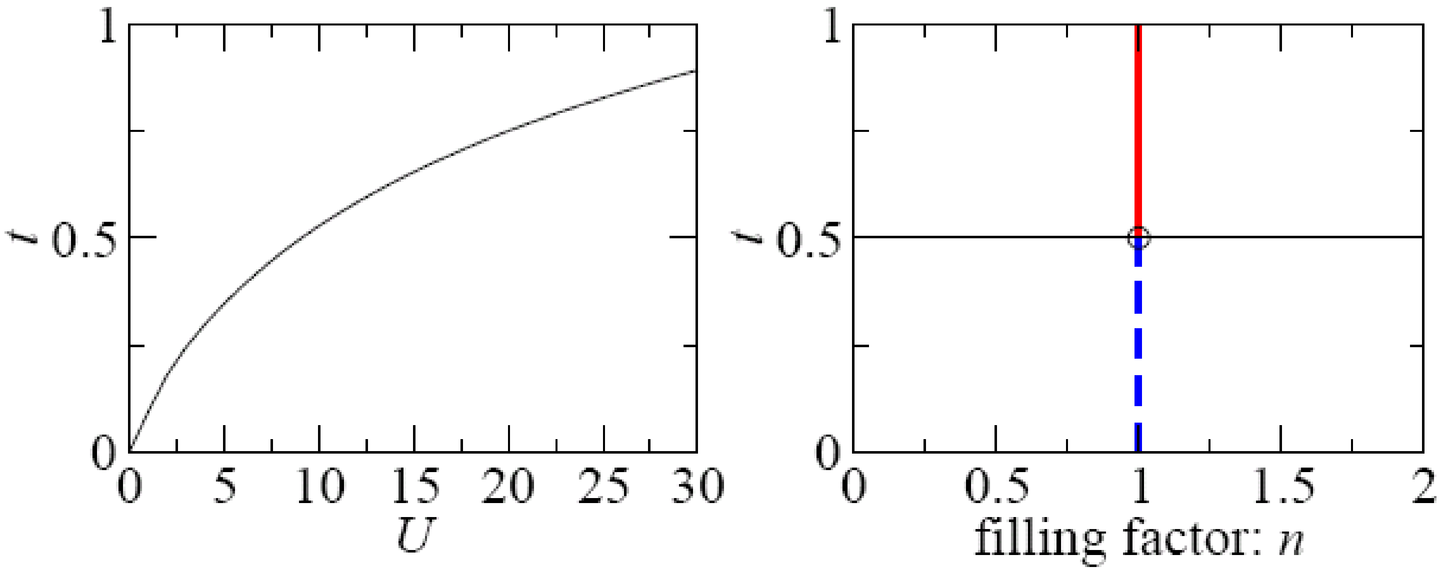


$$V_0 = \nu \frac{\hbar^2 k^2}{2m},$$

$$\frac{t_\beta}{t_\alpha} = \frac{m_\alpha}{m_\beta},$$

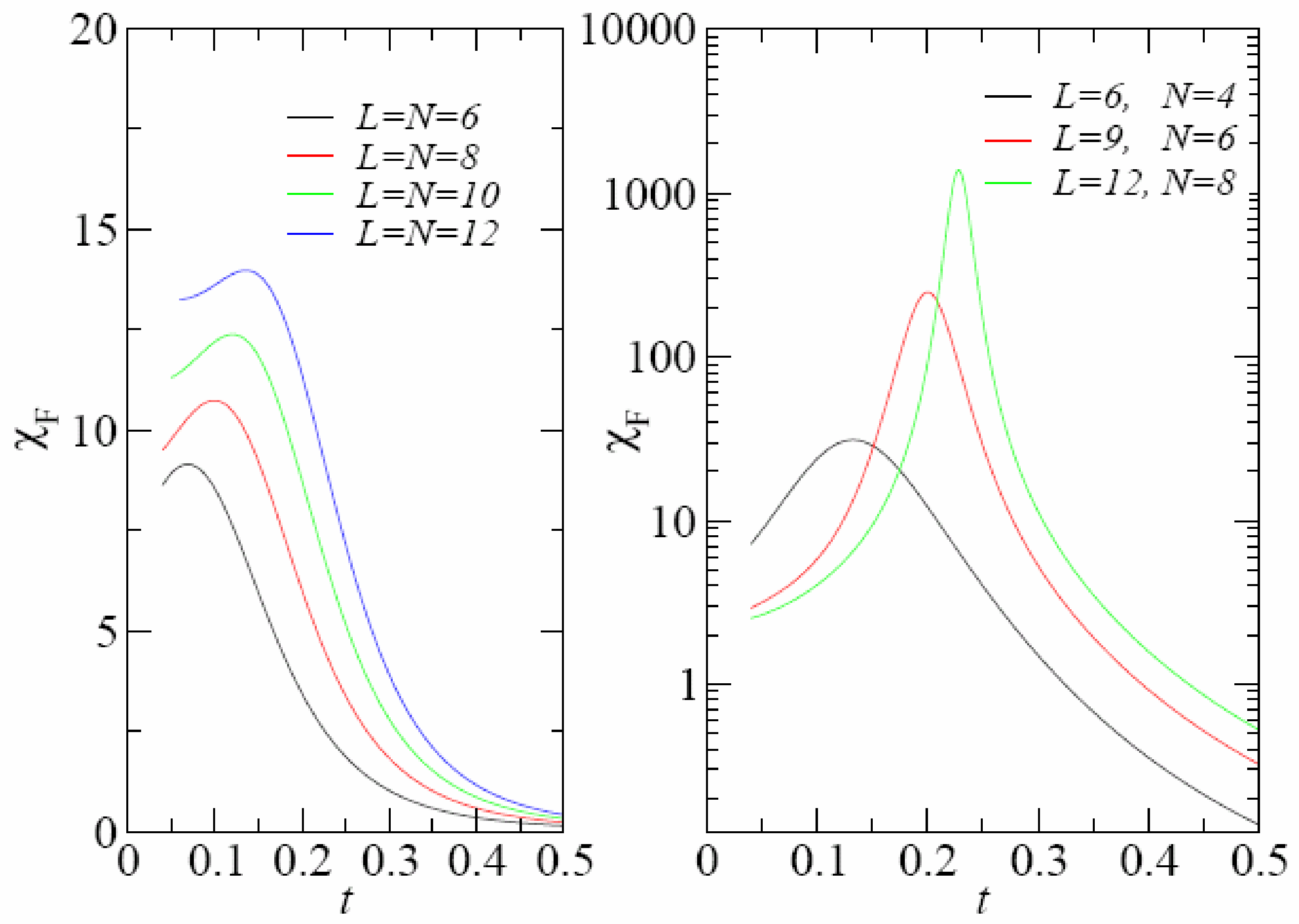
$$\frac{U}{t_\alpha} = \frac{16a\sqrt{\pi m_\alpha / m_\beta}}{\lambda} \frac{v^{1/4} v_\perp^{1/2}}{(\sqrt{v} + 2\sqrt{v_\perp})} \ e^{\pi^2 \sqrt{v}}$$

Application: asymmetric Hubbard model



The schematic phase diagram of the asymmetric Hubbard model

Application: asymmetric Hubbard model



Application: asymmetric Hubbard model

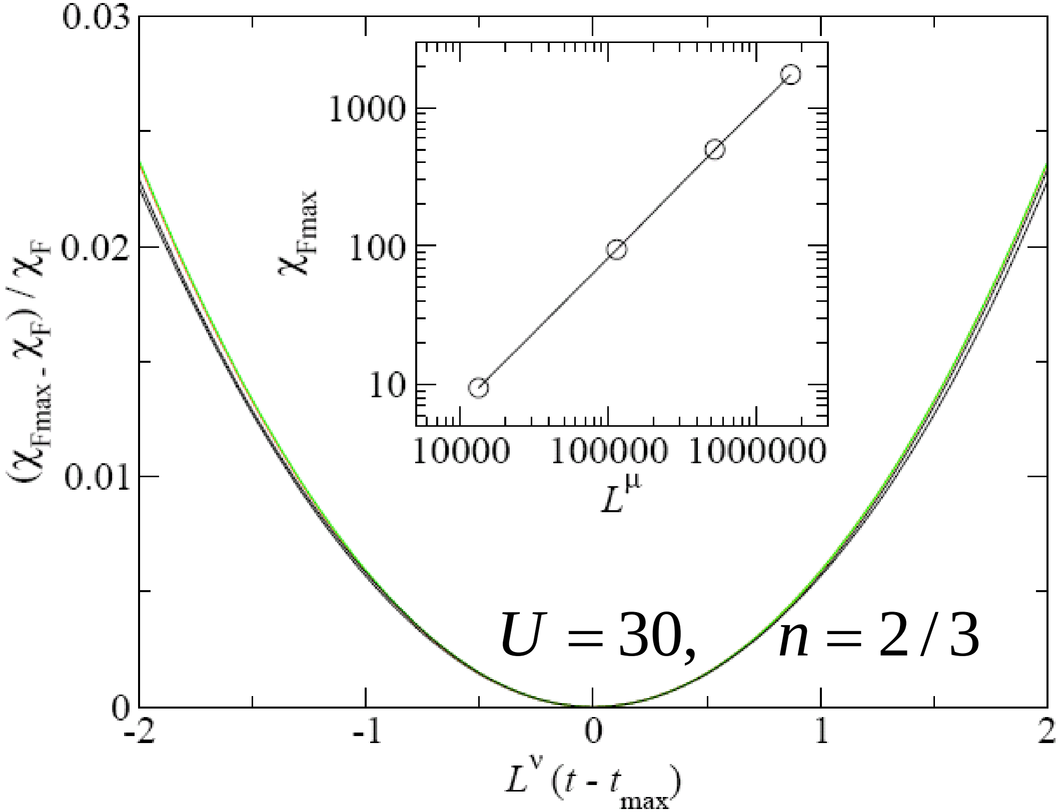
$$\frac{\chi_{F(t)}(t=t_{\max})-\chi_{F(t)}(t)}{\chi_{F(t)}(t)}=g[L^{\upsilon}(t-t_{\max})]$$

$$\mu = 5.3,$$

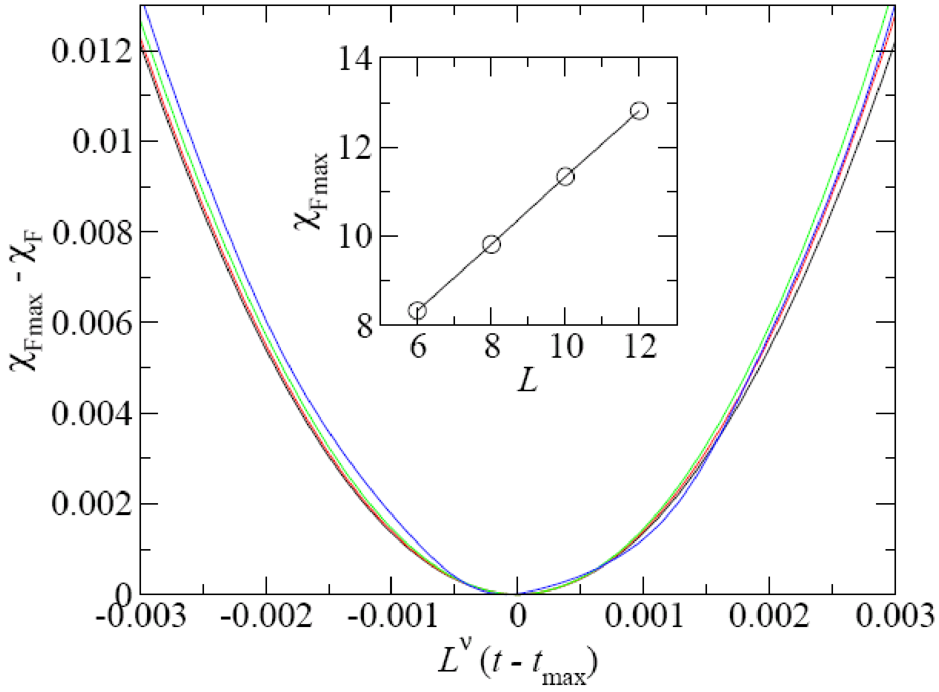
$$\nu = 2.65$$

$$\alpha = (\mu - 1)/\nu$$

$$= 0.6$$



Application: asymmetric Hubbard model



$$U = 30, \quad n = 1$$

$$\chi_{F(t)}(t) \simeq 3.855 + 0.7478L + 1349.9L^{-1/2}(t - t_{\max})^2$$

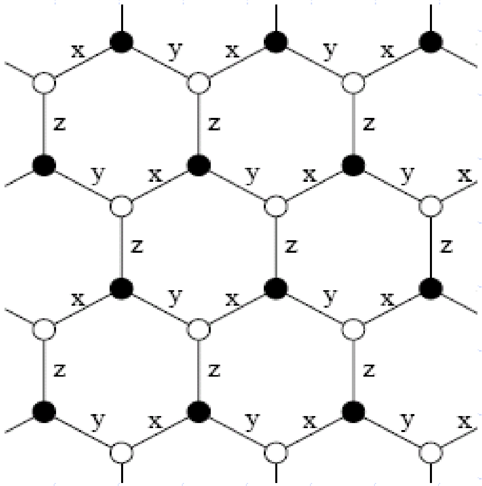
Content

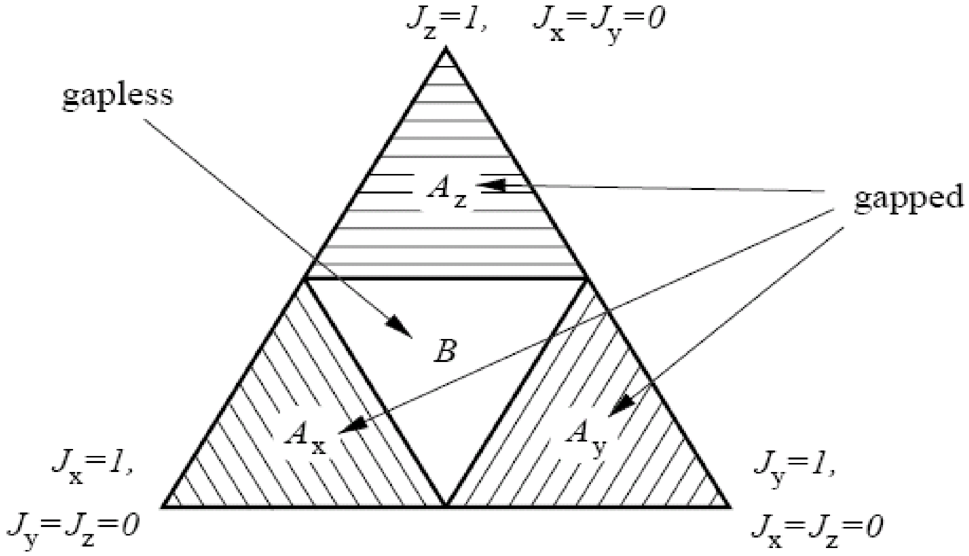
The Kitaev honeycomb model

- A. Kitaev, Ann. Phys. **303**, 2 (2003); Ann. Phys. **321**, 2 (2006).

$$H = -J_x \sum_{x\text{-bonds}} \sigma_j^x \sigma_k^x - J_y \sum_{y\text{-bonds}} \sigma_j^y \sigma_k^y - J_z \sum_{z\text{-bonds}} \sigma_j^z \sigma_k^z,$$

$$J_x + J_y + J_z = 1$$





X. G. Wen, *Quantum Field Theory of Many-Body Systems* (Oxford University, New York, 2004).

The critical fidelity susceptibility

$$\sigma^x = \mathrm{i}b^x c,\ \sigma^y = \mathrm{i}b^y c,\ \sigma^z = \mathrm{i}b^z c.$$

$$H = \frac{\mathrm{i}}{2}\sum_{j,k} \widehat{u}_{jk} J_{a_{jk}} c_j c_k$$

$$H = \sum_{\mathbf{q}} \begin{pmatrix} a_{-\mathbf{q},1} \\ a_{-\mathbf{q},2} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 0 & \mathrm{i}f(\mathbf{q}) \\ -\mathrm{i}f(\mathbf{q})^* & 0 \end{pmatrix} \begin{pmatrix} a_{\mathbf{q},1} \\ a_{\mathbf{q},2} \end{pmatrix}$$

$$H = \sum_{\mathbf{q}} \sqrt{\epsilon_{\mathbf{q}}^2 + \Delta_{\mathbf{q}}^2} \left(C_{\mathbf{q},1}^\dagger C_{\mathbf{q},1} - C_{\mathbf{q},2}^\dagger C_{\mathbf{q},2} \right)$$

$$\begin{aligned} f(\mathbf{q}) &= \epsilon_{\mathbf{q}} + \mathrm{i}\Delta_{\mathbf{q}}, \\ \epsilon_{\mathbf{q}} &= J_x \cos q_x + J_y \cos q_y + J_z \\ \Delta_{\mathbf{q}} &= J_x \sin q_x + J_y \sin q_y. \end{aligned}$$

The critical fidelity susceptibility

$$
\begin{aligned}
|\Psi_0\rangle &= \prod_{\mathbf{q}} C^{\dagger}_{\mathbf{q},2}\,|0\rangle \\
&= \prod_{\mathbf{q}} \frac{1}{\sqrt{2}} \left(\frac{\sqrt{\epsilon_{\mathbf{q}}^2 + \Delta_{\mathbf{q}}^2}}{\Delta_{\mathbf{q}} + i\epsilon_{\mathbf{q}}} a_{-\mathbf{q},1} + a_{-\mathbf{q},2} \right) |0\rangle
\end{aligned}
$$

$$E_0 = -\sum_{\mathbf{q}} \sqrt{\epsilon_{\mathbf{q}}^2 + \Delta_{\mathbf{q}}^2}.$$

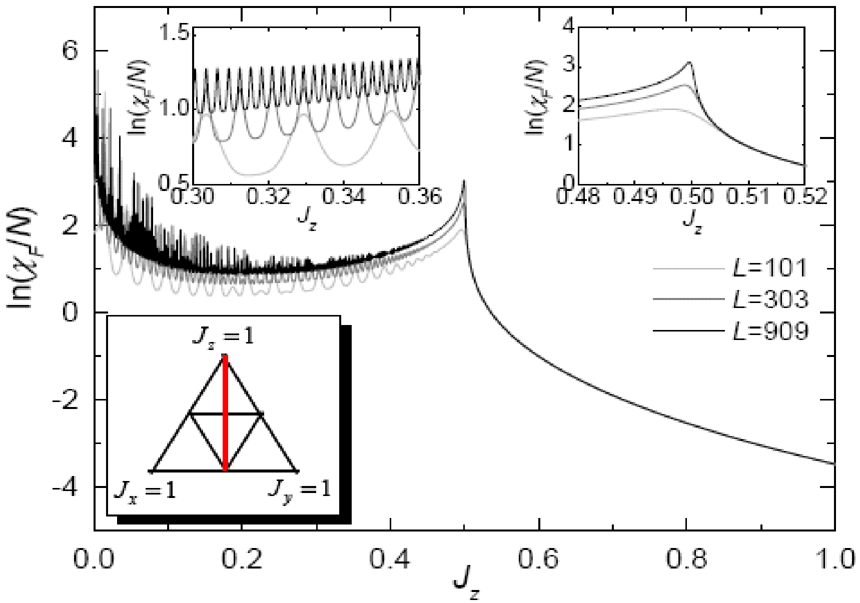
$$F^2 = \prod_{\mathbf{q}} \frac{1}{2} \left(1 + \frac{\Delta_{\mathbf{q}}\Delta'_{\mathbf{q}} + \epsilon_{\mathbf{q}}\epsilon'_{\mathbf{q}}}{E_{\mathbf{q}} E'_{\mathbf{q}}} \right)$$

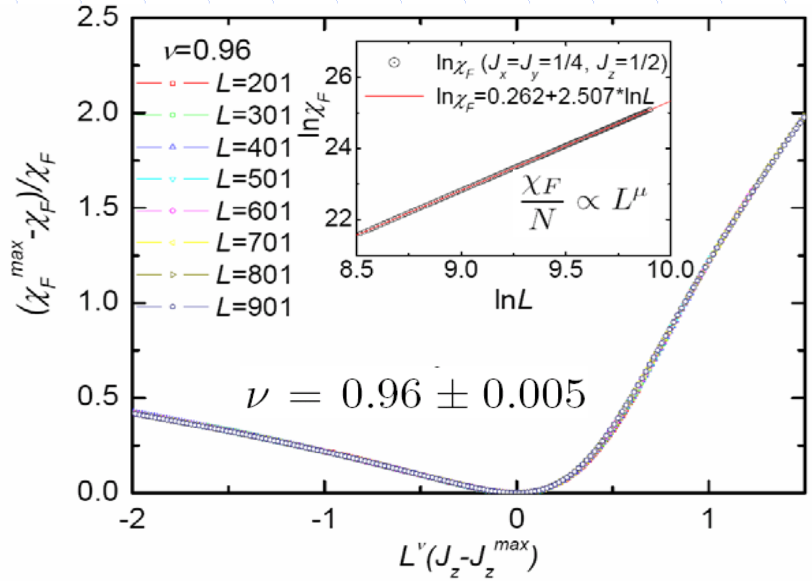
$$\chi_F \equiv \lim_{\delta\lambda \to 0} \frac{-2\ln F_i}{\delta\lambda^2}$$

The critical fidelity susceptibility

$$\chi_F = \frac{1}{16}\sum_{\mathbf{q}}\left[\frac{\sin q_x + \sin q_y}{\epsilon_{\mathbf{q}}^2 + \Delta_{\mathbf{q}}^2}\right]^2$$

$$\mu = 0.507 \pm 0.0001$$





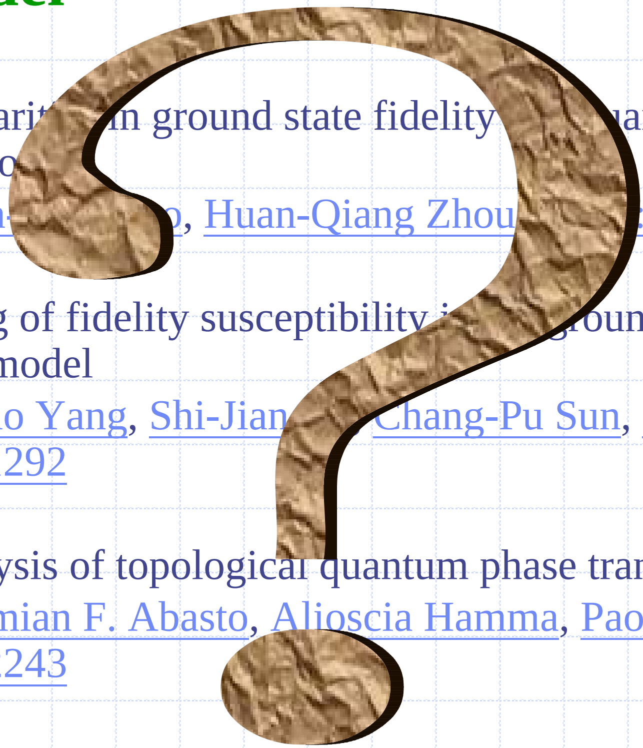
$$\nu = 0.96 \pm 0.005$$

$$\frac{\chi_F^{\max} - \chi_F}{\chi_F} = f[L^\nu (J_z - J_z^{\max})]$$

$$\frac{\chi_F}{N} \propto \frac{1}{|J_z - J_z^c|^\alpha} \qquad \alpha = \frac{\mu}{\nu} = 0.528 \pm 0.001$$

The first conclusion on topological phase transition

- **The fidelity susceptibility can be used to witness the topological quantum phase transition in the Kitaev model**



Title: Singularit[illegible] in ground state fidelity [illegible] quantum phase transitions for the Kitaev mo[illegible]
Authors: Jian-[illegible], Huan-Qiang Zhou [illegible]:0803.0814

Title: Scaling of fidelity susceptibility [illegible] ground state of Kitaev honeycomb model
Authors: Shuo Yang, Shi-Jian [illegible] Chang-Pu Sun, Hai-Qing Lin, arXiv:0803.1292

Fidelity analysis of topological quantum phase transitions
Authors: Damian F. Abasto, Alioscia Hamma, Paolo Zanardi, arXiv:0803.2243

The bond-bond correlation

Fidelity susceptibility W. L. You, Y. W. Li, and S. J. Gu, Phys. Rev. E **76**, 022101 (2007).

$$\chi_F = \int \tau \left[\langle \Psi_0 | H_I(\tau) H_I(0) | \Psi_0 \rangle - \langle \Psi_0 | H_I | \Psi_0 \rangle^2 \right] d\tau$$

Bond-bond correlation function

$$\begin{aligned} C(\mathbf{r}_1, \mathbf{r}_2) &= \left\langle \sigma^z_{\mathbf{r}_1,1} \sigma^z_{\mathbf{r}_1,2} \sigma^z_{\mathbf{r}_2,1} \sigma^z_{\mathbf{r}_2,2} \right\rangle \\ &\quad - \left\langle \sigma^z_{\mathbf{r}_1,1} \sigma^z_{\mathbf{r}_1,2} \right\rangle \left\langle \sigma^z_{\mathbf{r}_2,1} \sigma^z_{\mathbf{r}_2,2} \right\rangle \end{aligned}$$

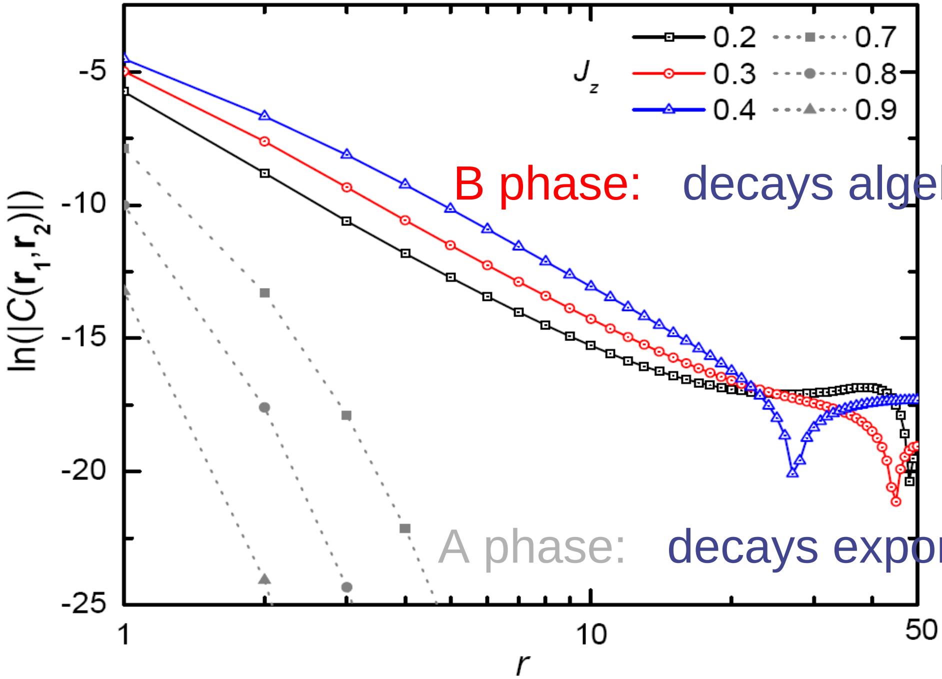
The bond-bond correlation

$$\left\langle \sigma^z_{\mathbf{r}_1,1}\sigma^z_{\mathbf{r}_1,2}\right\rangle = \left\langle \sigma^z_{\mathbf{r}_2,1}\sigma^z_{\mathbf{r}_2,2}\right\rangle = \frac{1}{N}\sum_{\mathbf{q}}\frac{\epsilon_{\mathbf{q}}}{E_{\mathbf{q}}}$$

$$\begin{aligned}
&\left\langle \Psi_0\right| \sigma^z_{\mathbf{r}_1,1}\sigma^z_{\mathbf{r}_1,2}\sigma^z_{\mathbf{r}_2,1}\sigma^z_{\mathbf{r}_2,2}\left|\Psi_0\right\rangle \\
= \; & \frac{1}{N^2}\sum_{\mathbf{q},\mathbf{q}'}\left\{\cos\left[\left(\mathbf{q}-\mathbf{q}'\right)\left(\mathbf{r}_1-\mathbf{r}_2\right)\right]-1\right\} \\
& \times \frac{\left(\Delta_{\mathbf{q}}\Delta_{\mathbf{q}'}-\epsilon_{\mathbf{q}}\epsilon_{\mathbf{q}'}\right)}{E_{\mathbf{q}}E_{\mathbf{q}'}}
\end{aligned}$$

$$C\left(\mathbf{r}_1,\mathbf{r}_2\right) \propto \frac{1}{\left|\mathbf{r}_1-\mathbf{r}_2\right|^4}$$

$$\frac{1}{\xi} = 2\sinh^{-1}\frac{\sqrt{2J_z-1}}{1-J_z}$$



B phase: decays algebraically

A phase: decays exponentially

Content

Beyond the pure-state fidelity

1. Fidelity per site H. Q. Zhou, J. H. Zhao, and B. Li, arXiv:0704.2940;

2. Operator fidelity X. Wang, Z. Sun, and Z. D. Wang, arXiv:0803.2940

3. Density-functional fidelity (SJGu, preprint)

4. Reduced fidelity (Zhou. Peres. Wang. GU)

The density-functional theory

$$\hat{H}(\lambda) = \hat{H}_0 + \lambda \hat{H}_I + \sum_x \mu_x \hat{n}_x$$

the density-functional fidelity (DFF). According to the Hohenberg-Kohn theorems, the ground-state properties of a quantum many-body system is uniquely determined by the density distribution n_x that minimize the functional for the ground-state energy $E_0[n_x]$. Therefore, the distribution n_x captures the most relevant information about the ground-state. Any change in the structure of the wavefunction can be found by calculating the similarity between two density distributions, i.e. the fidelity,

The density-functional fidelity

$$\rho(x_1, x_1') = \mathrm{tr}|\Psi_0(\lambda)\rangle\langle\Psi_0(\lambda)|$$

$$n = \sum_x n_x |x\rangle\langle x|$$

$$n_x = \langle\Psi_0(\lambda)|\hat{n}_x|\Psi_0(\lambda)\rangle = \langle\Psi_0(\lambda)|(\partial\hat{H}/\partial\mu_x)|\Psi_0(\lambda)\rangle$$

$$F(\lambda, \lambda\prime) = \mathrm{tr}\sqrt{n(\lambda)n(\lambda')}$$

The density-functional theory

$$F(\lambda, \lambda\prime) = \mathrm{tr}\sqrt{n(\lambda)n(\lambda')}$$

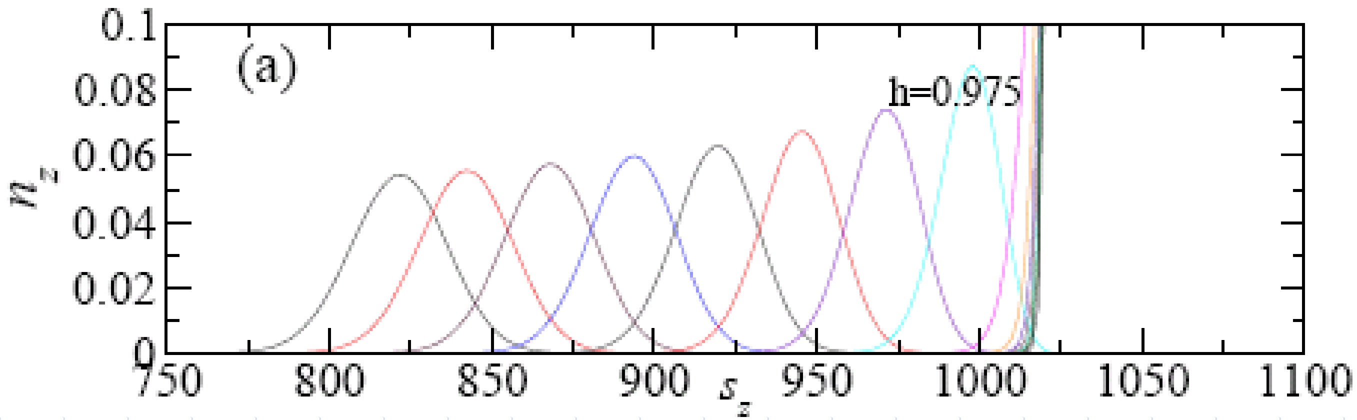
$$F(\lambda, \lambda + \delta\lambda) = 1 - \frac{(\delta\lambda)^2}{2}\chi_F.$$

$$\chi_F = \sum_x \frac{1}{4n_x}\left(\frac{\partial n_x}{\partial \lambda}\right)^2$$

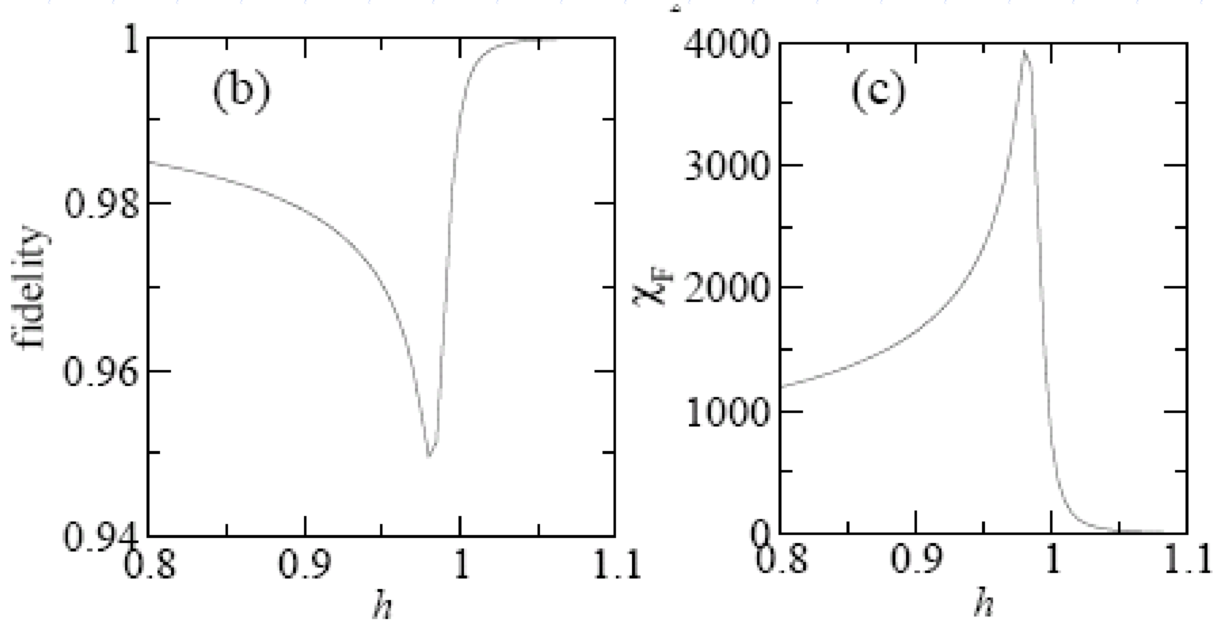
Example: the LMG model

$$H = -\frac{\lambda}{2S}(1+\gamma)\left(\hat{\mathbf{S}}^2 - \hat{S}_z^2 - S\right) - 2h\hat{S}_z$$
$$-\frac{\lambda}{4S}(1-\gamma)\left(\hat{S}_+^2 + \hat{S}_-^2\right),$$

$$|\Psi_0(h)\rangle = \sum \varphi(s_z)|s_z\rangle \qquad n_z = \langle\Psi_0(h)|\hat{s}_z|\Psi_0(h)\rangle$$



Example: the LMG model



For the LMG model, the density-functional fidelity is the same as the pure-state fidelity because the LMG model become a single-particle problem in the anisotropic case.

Example: Hubbard model

$$H = -\sum_{\sigma,j}(c_{j,\sigma}^{+}c_{j+1,\sigma} + c_{j+1,\sigma}^{+}c_{j,\sigma}) + U\sum_{j} n_{j,\uparrow}n_{j,\downarrow}$$

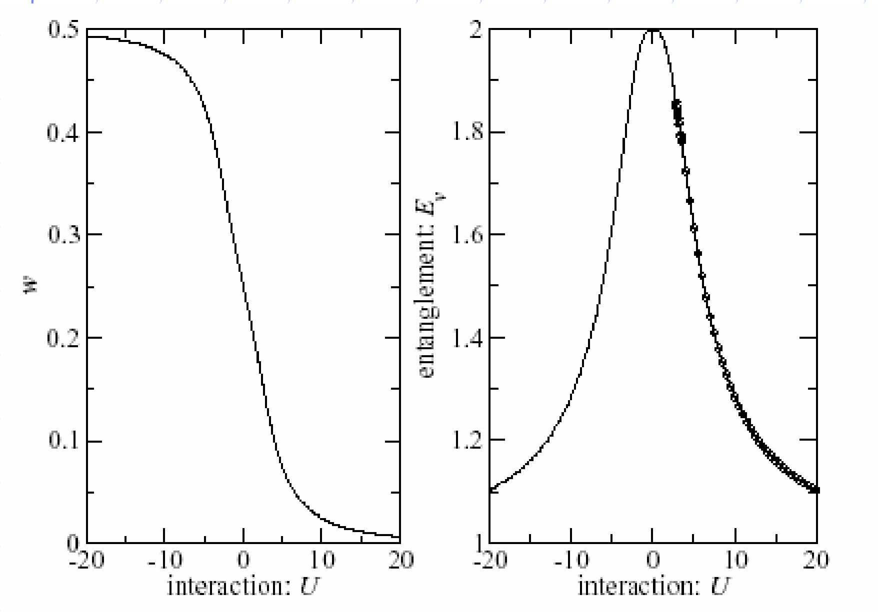
$$\boxed{\begin{aligned} 2\pi I_j &= k_j L - 2\sum_{a=1}^{M}\tan^{-1}\left(\frac{\lambda_a - \sin k_j}{U/4}\right) \\ 2\pi J_a &= 2\sum_{j=1}^{N}\tan^{-1}\left(\frac{\lambda_a - \sin k_j}{U/4}\right) - 2\sum_{b=1}^{M}\tan^{-1}\left(\frac{\lambda_a - \lambda_b}{U/2}\right) \end{aligned}}$$

$$E = -2\sum_{j=1}^{N}\cos k_j$$

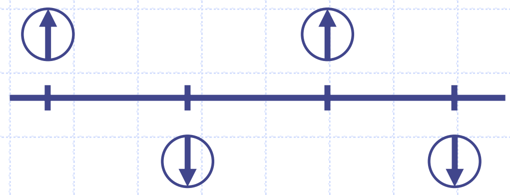
$$I_j = -\frac{N-1}{2}, -\frac{N-3}{2}, \cdots, \frac{N-1}{2} \qquad J_a = -\frac{M-1}{2}, -\frac{M-3}{2}, \cdots, \frac{M-1}{2}$$

Hellman-Feynman定理: $w = \frac{1}{N}\frac{dE(U)}{dU}$

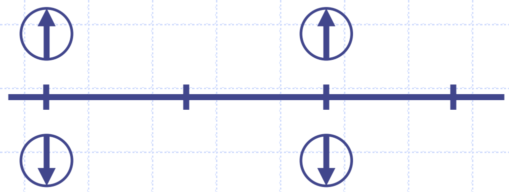
Example: Hubbard model



$$U > 0$$



$$U < 0$$

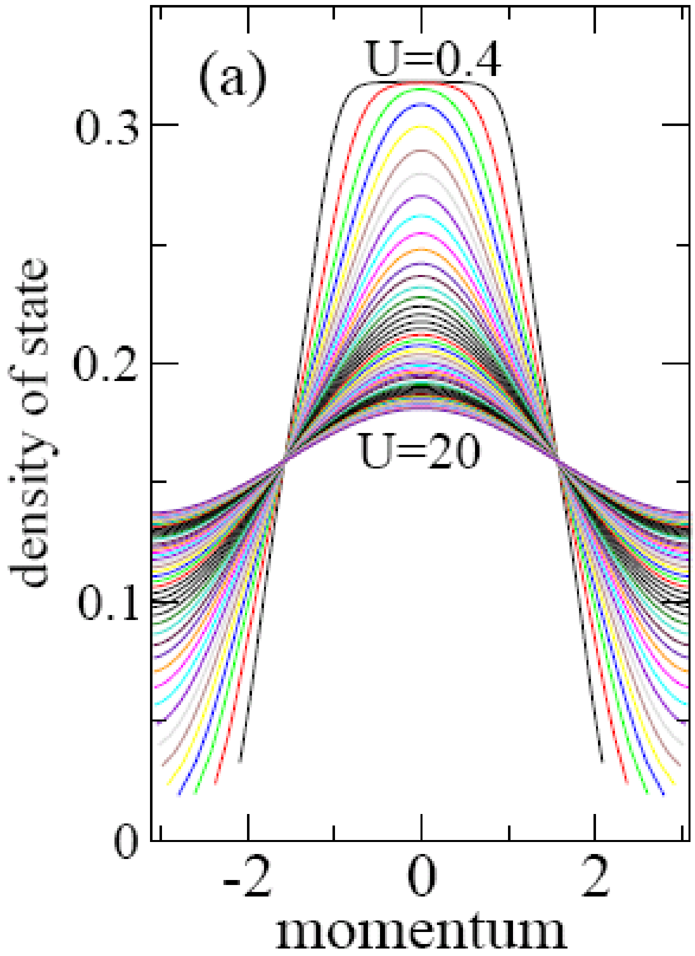


$$w = \int_0^{\infty} \frac{J_0(\omega) J_1(\omega) d\omega}{1 + \cosh(U\omega / 2)}$$

$$E_v(U) = E_v(-U)$$

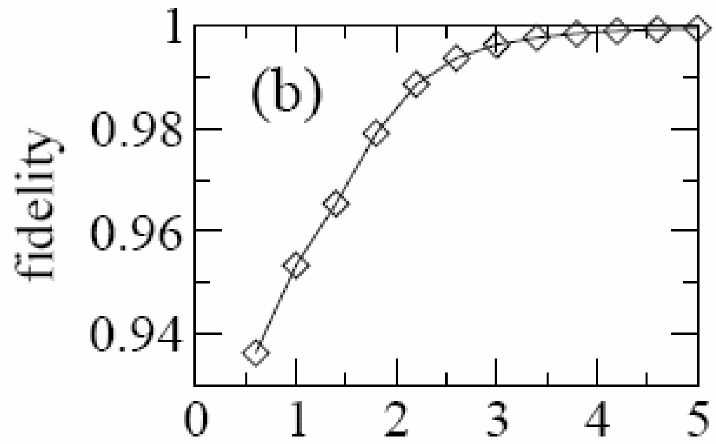
$$E_v = -2w \log_2 w - 2(1/2 - w) \log_2 (1/2 - w)$$

Example: Hubbard model

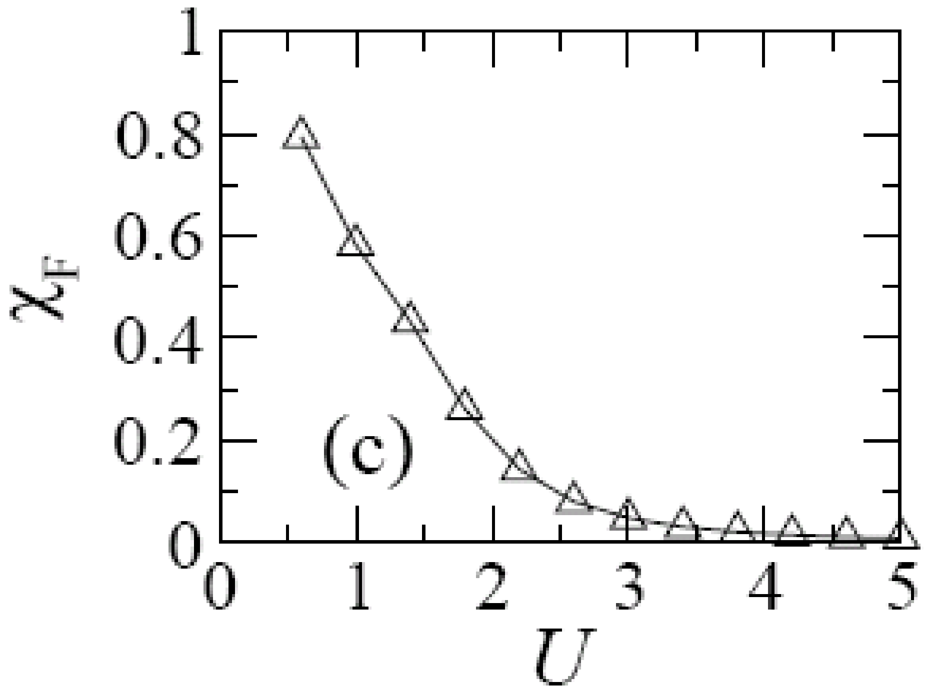


$$\rho\left(\frac{k_j + k_{j+1}}{2}\right) = \frac{1}{L(k_{j+1} + k_j)}$$

$$\frac{N}{L} = \int \rho(k) dk$$



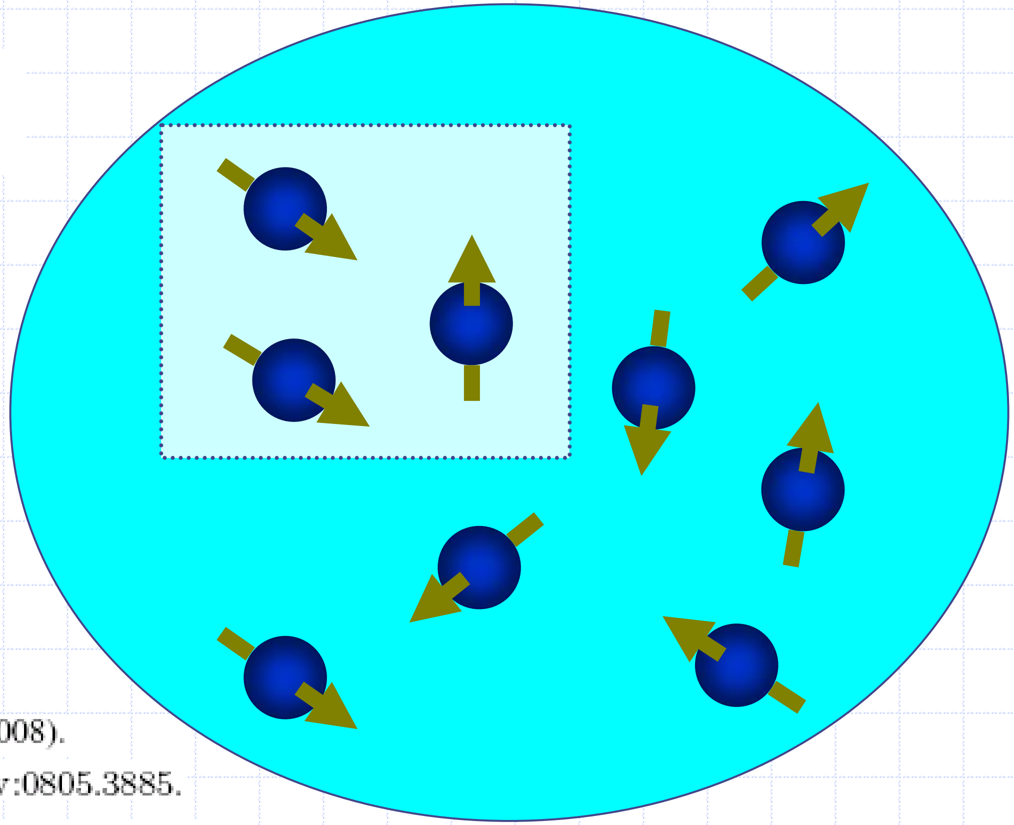
Example: Hubbard model



Example: Reduced fidelity

$$F_A\left(h,\tilde{h}\right) = \mathrm{Tr}\sqrt{\sqrt{\hat{\rho}_A(h)}\hat{\rho}_A(\tilde{h})\sqrt{\hat{\rho}_A(h)}}.$$

$$\rho_A(h) = \frac{1}{2}\begin{pmatrix} 1+\langle\sigma^z\rangle & 0 \\ 0 & 1-\langle\sigma^z\rangle \end{pmatrix}.$$



H. Q. Zhou, arXiv:0704.2945.

N. Paunkovic *et al*, Phys. Rev. A **77**, 052302 (2008).

H. M. Kwok, C. S. Ho, and S. J. Gu, arXiv:0805.3885.

J. Ma, L. Xu, H. Xiong, and X. Wang, arXiv:0805.4062.

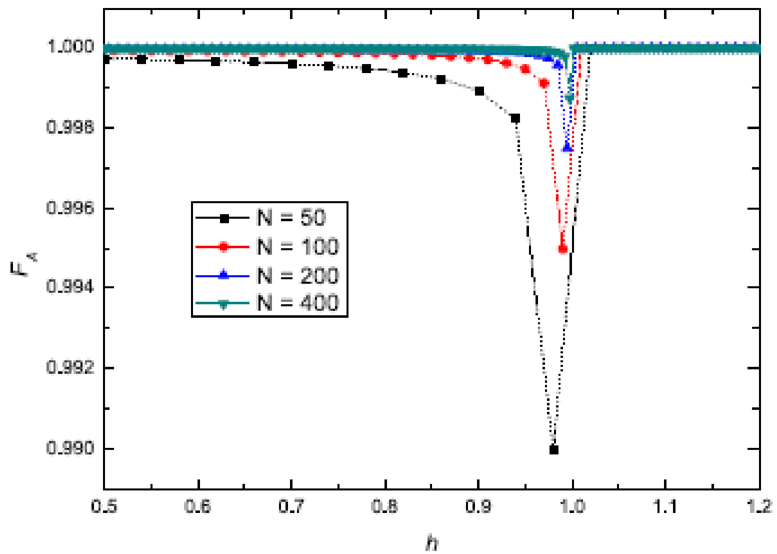
J. Ma, L. Xu, and X. Wang, arXiv:0808.1816.

H. N. Xiong, J. Ma, Z. Sun, and X. Wang, arXiv:0808.1817.

Example: Reduced fidelity

$$
\begin{aligned}
H_{\mathrm{LMG}} &= -\frac{1}{N}\sum_{i<j}\left(\sigma_x^i\sigma_x^j+\sigma_y^i\sigma_y^j\right)-h\sum_i\sigma_z^i \\
&= -\frac{2}{N}\left(S_x^2+S_y^2\right)-2hS_z+\frac{1}{2} \\
&= -\frac{2}{N}\left(\mathbf{S}^2-S_z^2-N/2\right)-2hS_z.
\end{aligned}
$$

$$E(M,h)=\frac{2}{N}\left(M-\frac{hN}{2}\right)^2-\frac{N}{2}\left(1+h^2\right).$$



Summary

1. We establish a general relation between the fidelity and dynamic structure factor of the driving parameter
2. We can learn the universality class of the critical phenomena from the Fidelity susceptibility.
3. Fidelity susceptibility and bond-bond long range correlation can also describe the topological phase transitions.
4. We propose a density-functional fidelity and use reduced fidelity to study the quantum phase transitions