Quantum Tunneling Effect of Topological Order and Topological Quantum Computation

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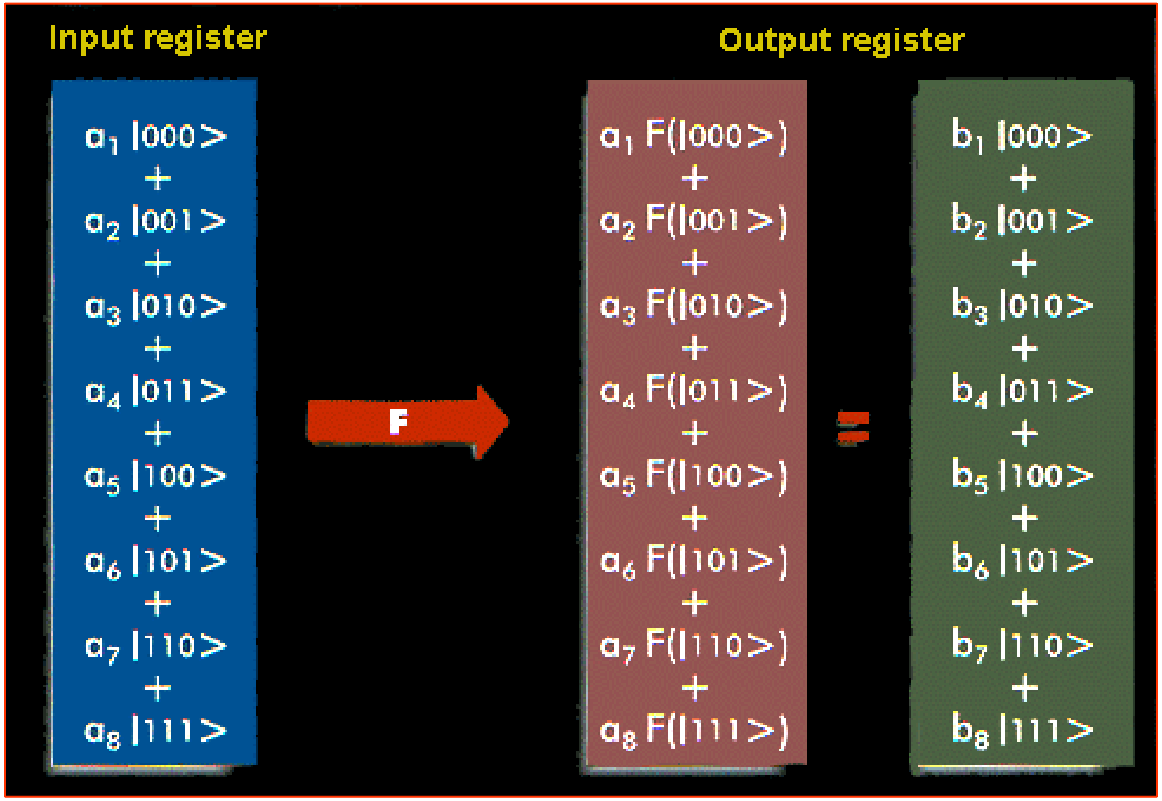
Outline

- **Quantum computation and topological quantum computation (TQC)**
- **Z2 topological order and topological degeneracy**
- **Quantum tunneling effect of topological order**
- **TQC from quantum tunneling effect**

I. Quantum Computing

- **Quantum computers are predicted to utilize quantum states to perform memory and to process tasks.**

Why Is This Helpful?

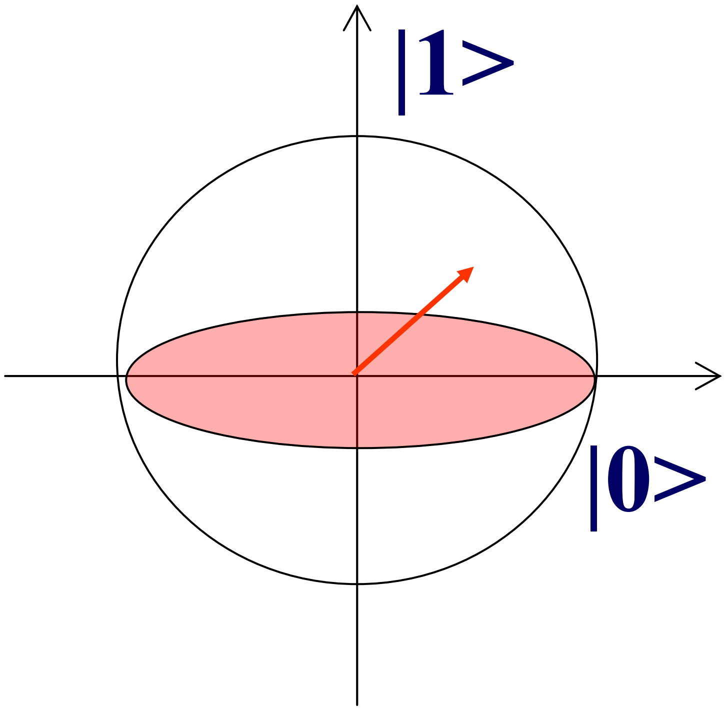


- Multiple computations simultaneously
- Computing power is exponential

Five criteria for physical implementation of a quantum computer - D. P. DiVincenzo

- **Well defined extendible qubit - stable memory**
- **Preparable in the “000…” state**
- **Long decoherence time (>10^4 operation time)**
- **Universal set of gate operations**
- **Single-quantum measurements**

Quantum bit - Qubit

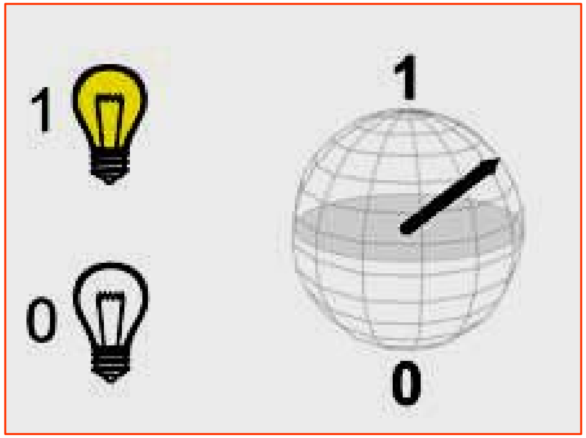


- Basis states |0>, |1>
- Arbitrary state:

$$\alpha|0\rangle+\beta|1\rangle,$$

α, β complex,

$$|\alpha|^2+|\beta|^2=1.$$



Physical qubits

- **Nuclear spin = orientation of atom's nucleus in magnetic field: $\uparrow = |0\rangle$, $\downarrow = |1\rangle$.**

- **Photons in a cavity:**
  No photon = $|0\rangle$, one photon = $|1\rangle$

- **...**

Quantum Logic Gates

An arbitrary unitary matrix may be decomposed as

$$e^{i\alpha}\begin{bmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{bmatrix}\begin{bmatrix} \cos\frac{\nu}{2} & -\sin\frac{\nu}{2} \\ \sin\frac{\gamma}{2} & \cos\frac{\nu}{2} \end{bmatrix}\begin{bmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2} \end{bmatrix}$$

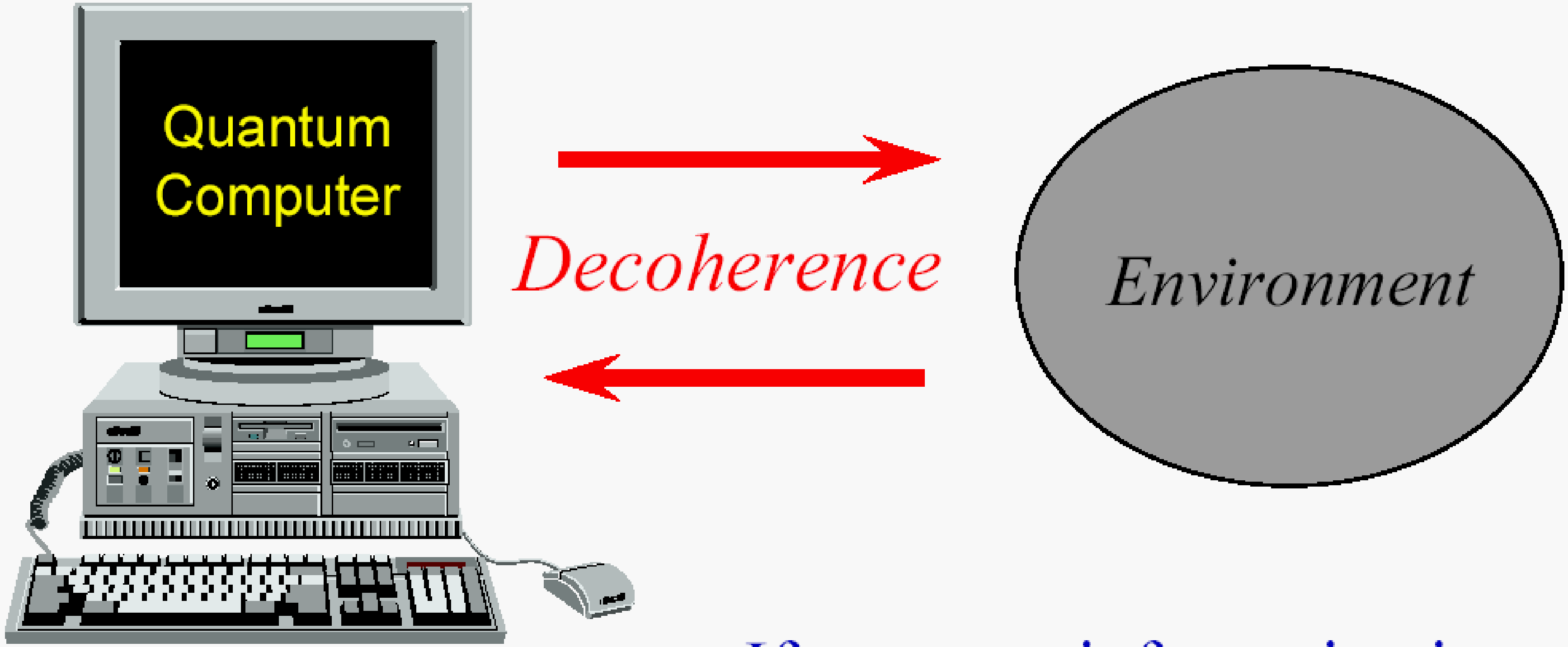
where α **,** β **,** ν **, and** δ **are real-valued.**

Frequently Used Gates

- **Hadamard:** **H** $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
- **Pauli-X:** **X** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- **Pauli-Y:** **Y** $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
- **Pauli-Z:** **Z** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- **Phase:** **S** $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
- **π /8:** **T** $\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$

Physical systems actively considered for quantum computer implementation

- Liquid-state NMR
- NMR spin lattices
- Linear ion-trap spectroscopy
- Neutral-atom optical lattices
- Cavity QED + atoms
- Linear optics with single photons
- Nitrogen vacancies in diamond
- Electrons on liquid He
- Josephson junctions arrays
- Spin spectroscopies, impurities in semiconductors
- Coupled quantum dots
- …



If quantum information is cleverly encoded, it *can* be protected from decoherence and other potential sources of error. Intricate quantum systems *can* be accurately controlled.

Solution?

- Since the topological properties is not changed by actions such as stretching, squashing and bending but not by cutting or joining, it prevents small perturbations from the environment.

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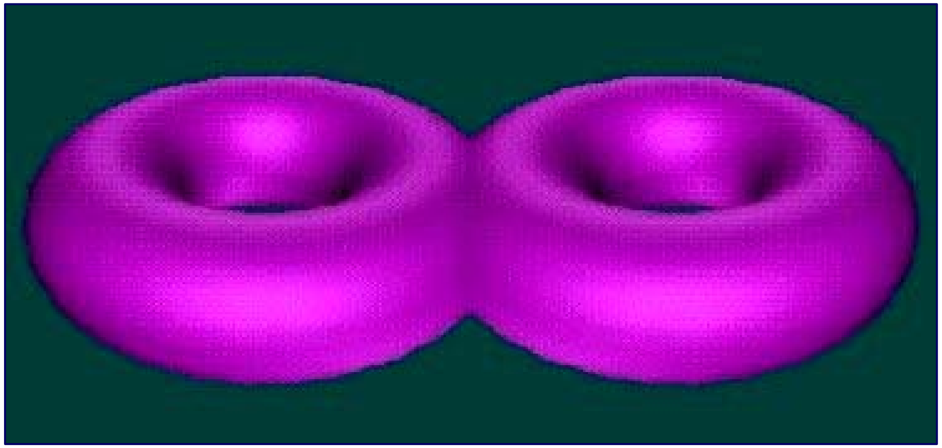
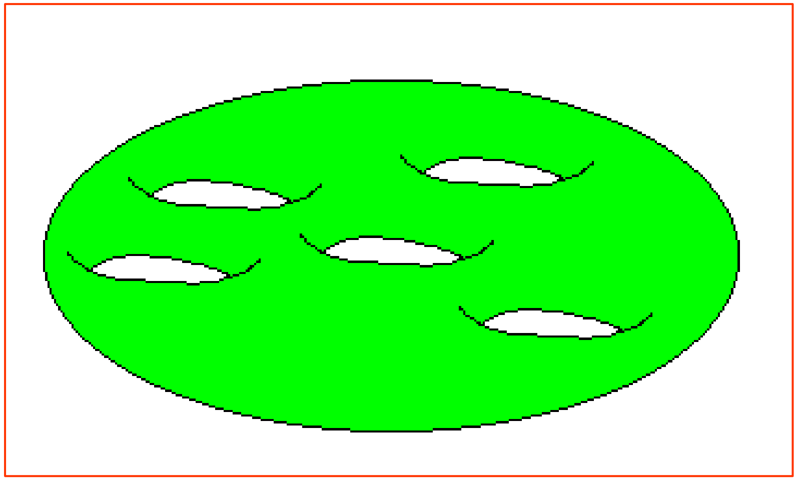
Genus

- The number of holes is called "genus" in topology.



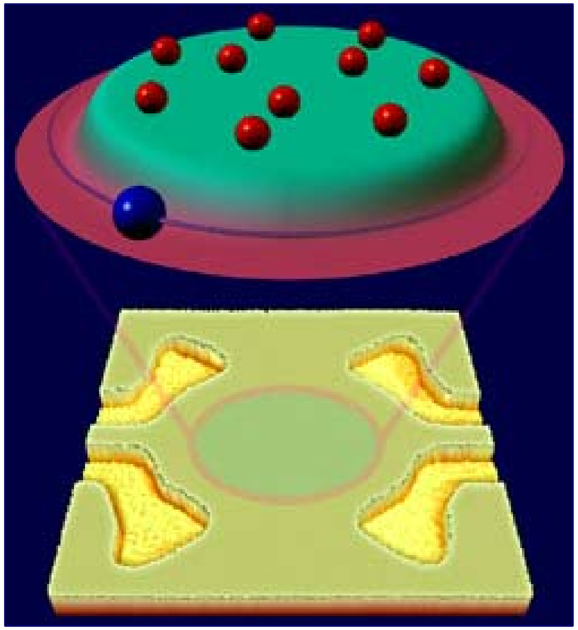


Pictures courtesy to http://www.math.toronto.edu/~drorbn/People/Eldar/thesis/SurAndHome.htm

Topological Quantum Computer

Anyon

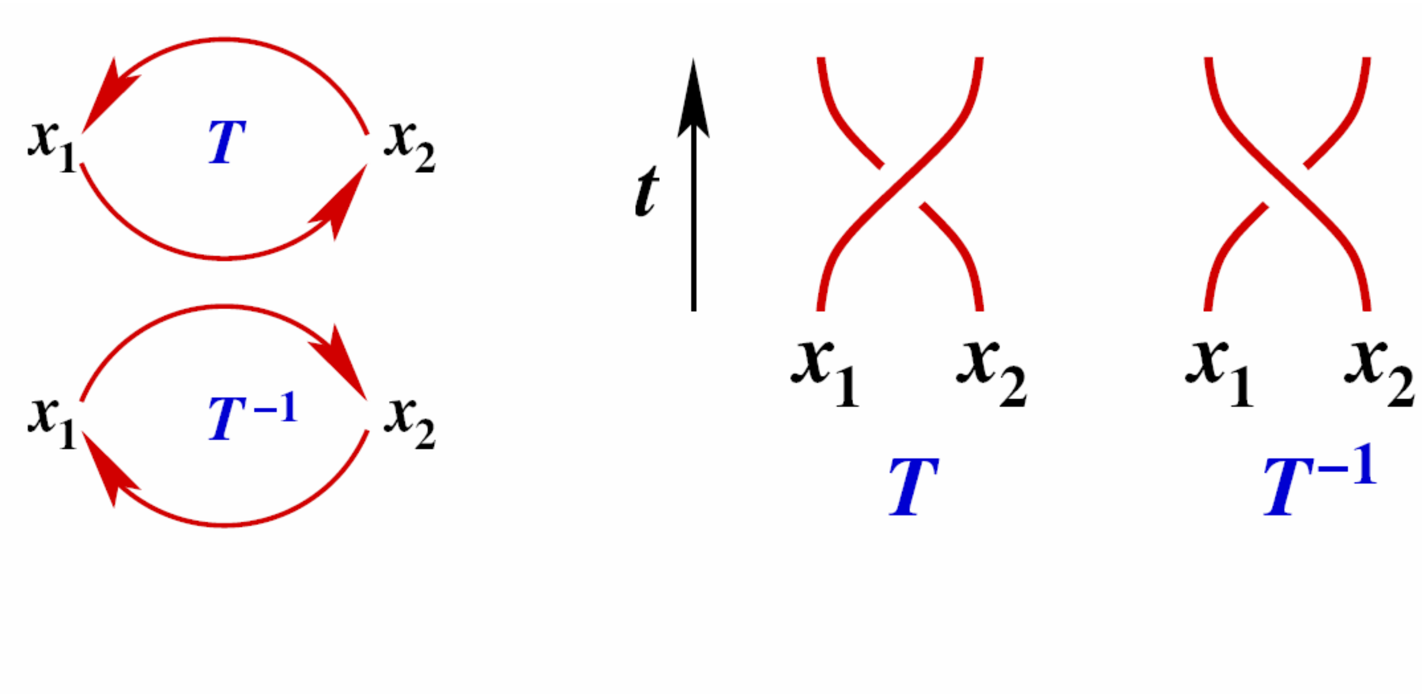
- Ordinarily, every particle in quantum theory is neatly classified as either a boson--a particle happy to fraternize with any number of identical particles in a single quantum state--or a fermion.
- Frank Wilczek used the term anyons in 1982 to describe such particles, since they can have "any" phase when particles are interchanged.



Fractional quantum Hall states

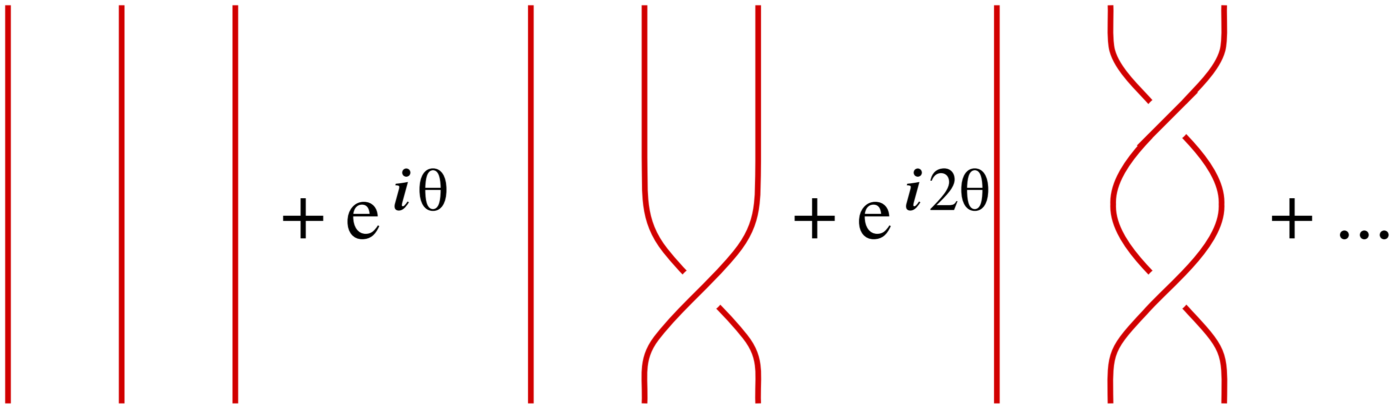
Courtesy to http://focus.aps.org/story/v16/st14

Exchange statistics in (2+1)D



- In (3+1)D, $T^{-1} = T$, while in (2+1)D $T^{-1} \neq T$!
- If $T^{-1} = T$ then $T^2 = 1$, and the only types of particles are bosons and fermions.

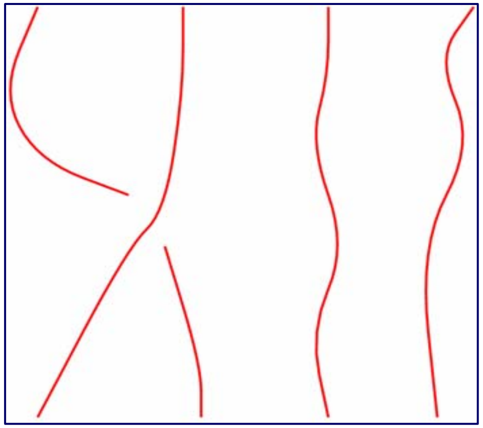
Abelian Anyons



- **These phase factors realise an Abelian anyon.**
- **Fractional Quantum Hall state at a filling factor $\nu = 1/m$.**
- **The ground state degeneracy on a torus: *m*-fold degenerate ground states for FQHE (Haldane, Rezayi '88, Wen '90).**

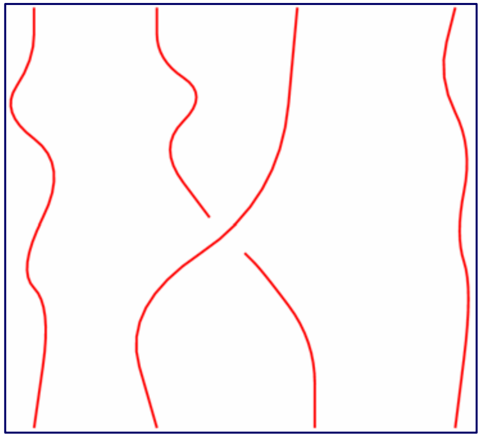
Non-Abelian Anyons

Exchanging particles 1 and 2:



$$\psi_a \rightarrow M^{12}_{ab}\psi_b$$

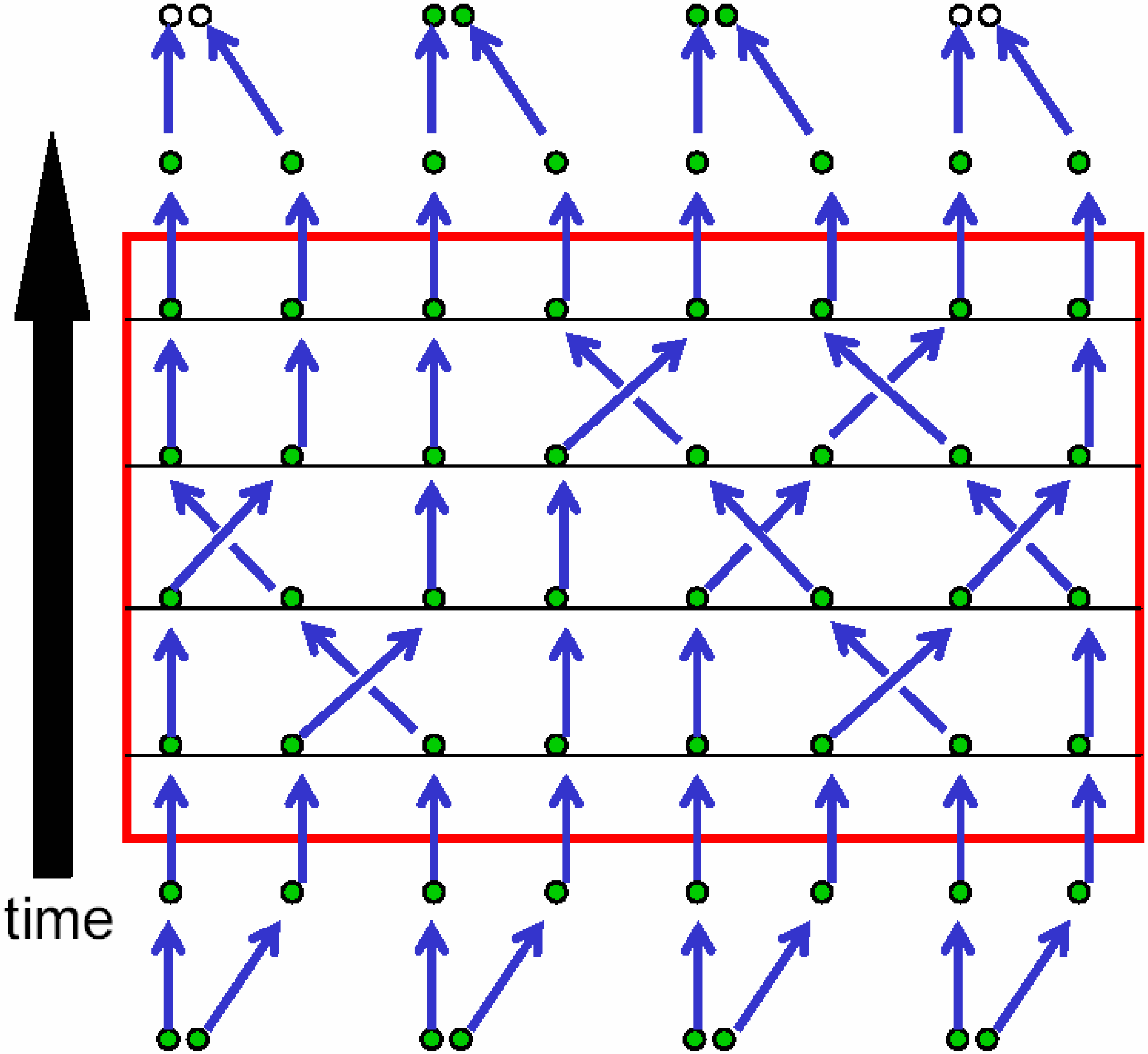
Exchanging particles 2 and 3:



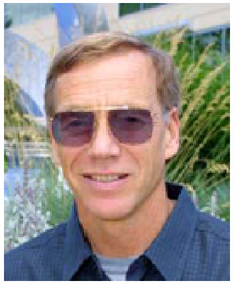
$$\psi_a \rightarrow M^{23}_{ab}\psi_b$$

- **Matrices M^{12} and M^{23} need not commute, hence Non-Abelian Statistics.**
- **Matrices M form a higher-dimensional representation of the braid-group.**
- **For fixed particle positions, we have a non-trivial multi-dimensional Hilbert space where we can store information.**

Topological quantum computation (Kitaev '97, FLW '00)



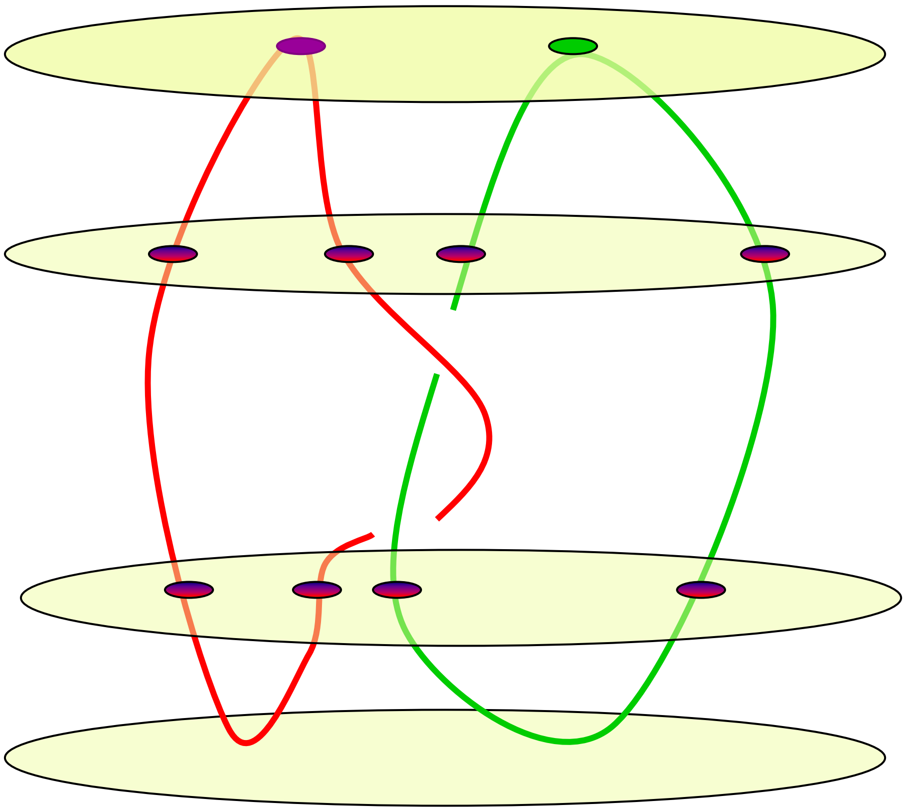
Kitaev



Freedman

Topological Quantum Computation

Computation	Physics
output	measure
apply operators	braid
initialize	create particles

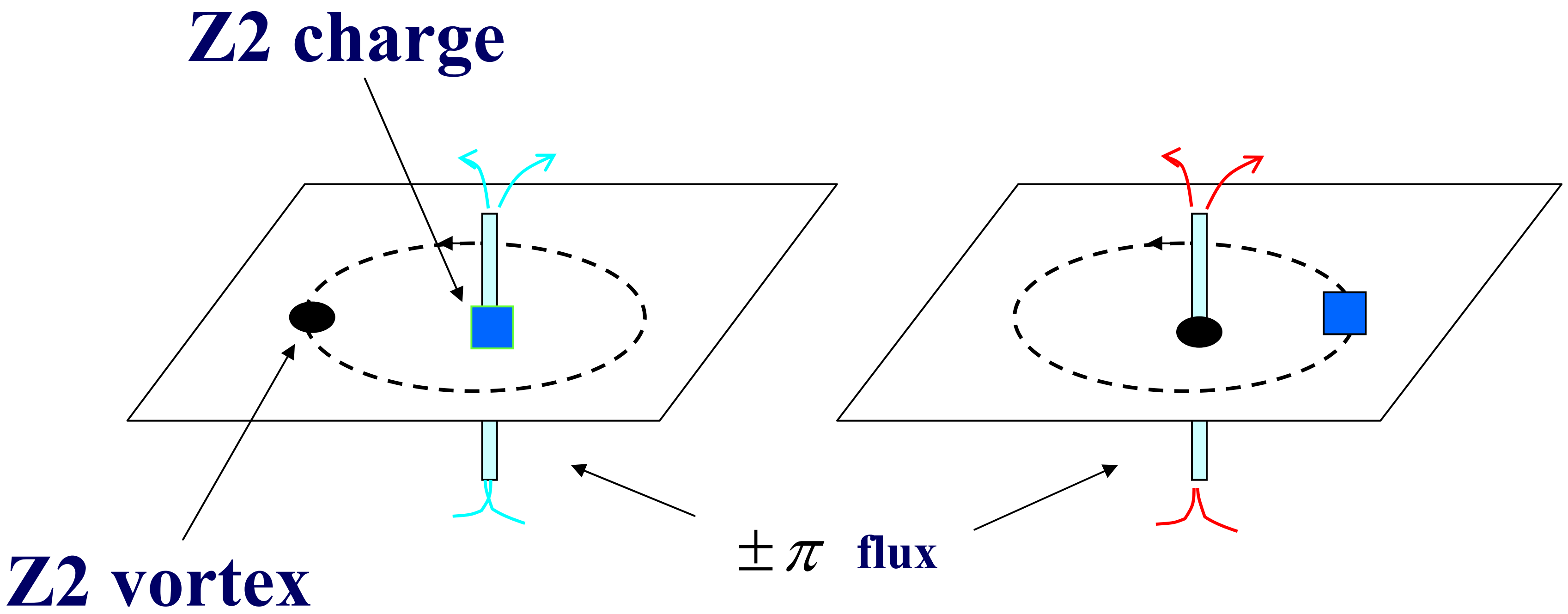


II、Introduction to Z2 topological order

2D Z2 topological orders are simplest topological state with the following three key properties :

- **All excitations are gapped**
- **Stable against all perturbation**
- **Topological degeneracy**
- ***Mutual semion* statistics.**

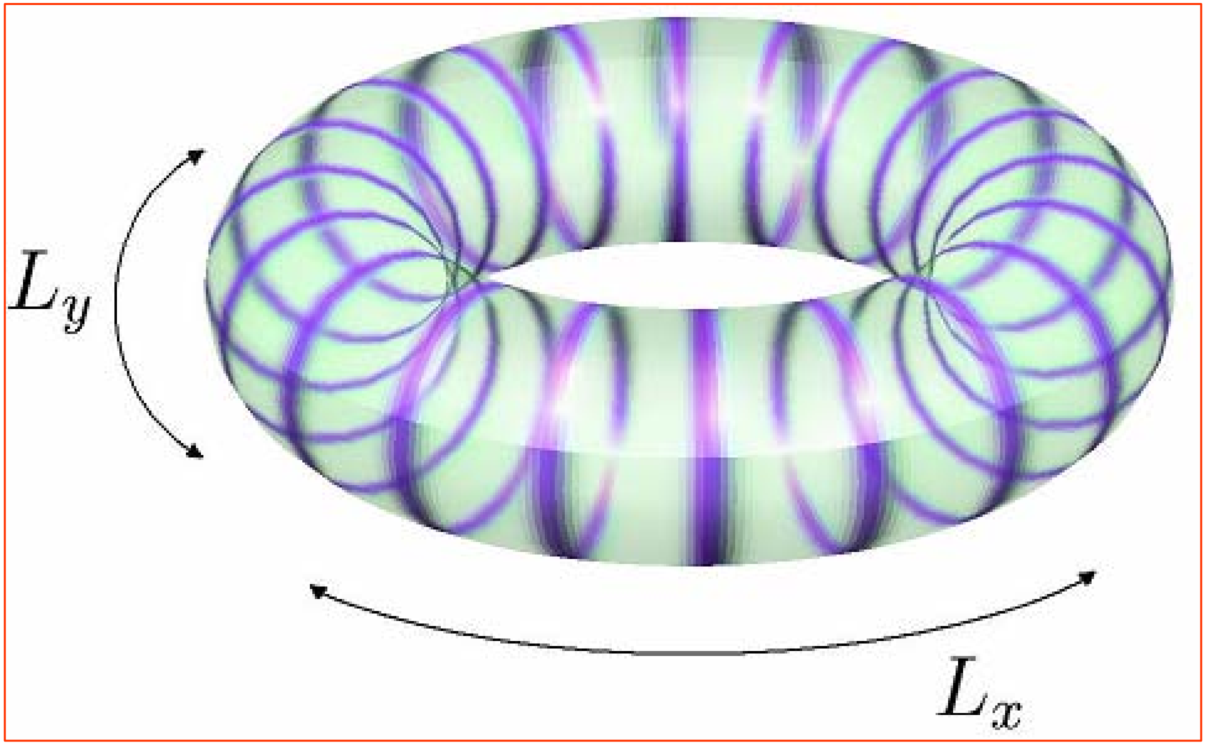
X. G. Wen PRB, **65**, 165113 (2002).



Mutual semion statistics

Topological degeneracy of Z2 topological orders

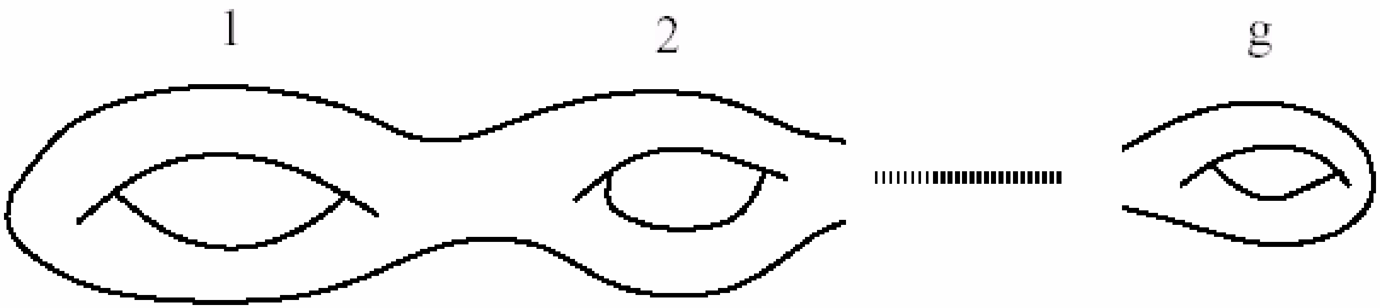
- **Topological degeneracy on a torus is always 4**



- **The topological degeneracy for the Z2 topological orders is determined by the genus of the Riemann surface,**

$$N = 4^g$$

- **g is the genus**



Topological qubit

Topological qubit

A. Yu. Kitaev, Annals Phys. 303, 2 (2003) [quant-ph/9707021]

|0> and |1> are the ground-states of a system

which are degenerate because of the (non-trivial) topology.

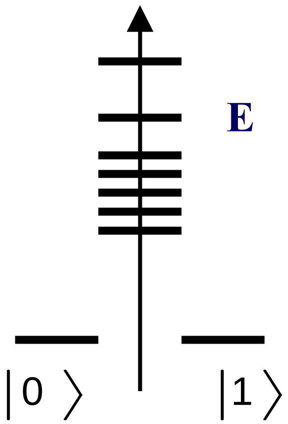
$$\left|\Psi\right\rangle = \alpha\left|0\right\rangle + \beta\left|1\right\rangle$$

Advantage

The two states are *locally* indistinguishable

$\Rightarrow$ no local perturbation can introduce decoherence.

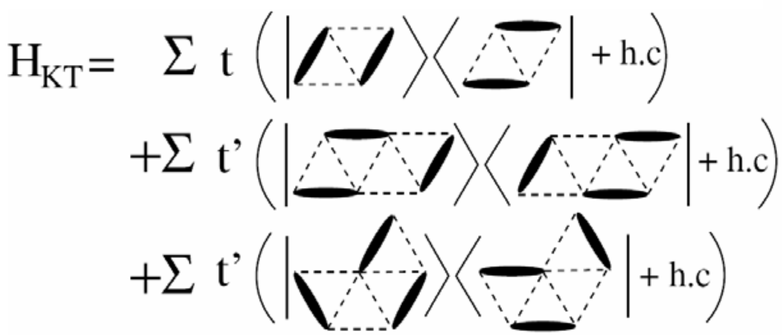
Ioffe, &, Nature 415, 503 (2002).

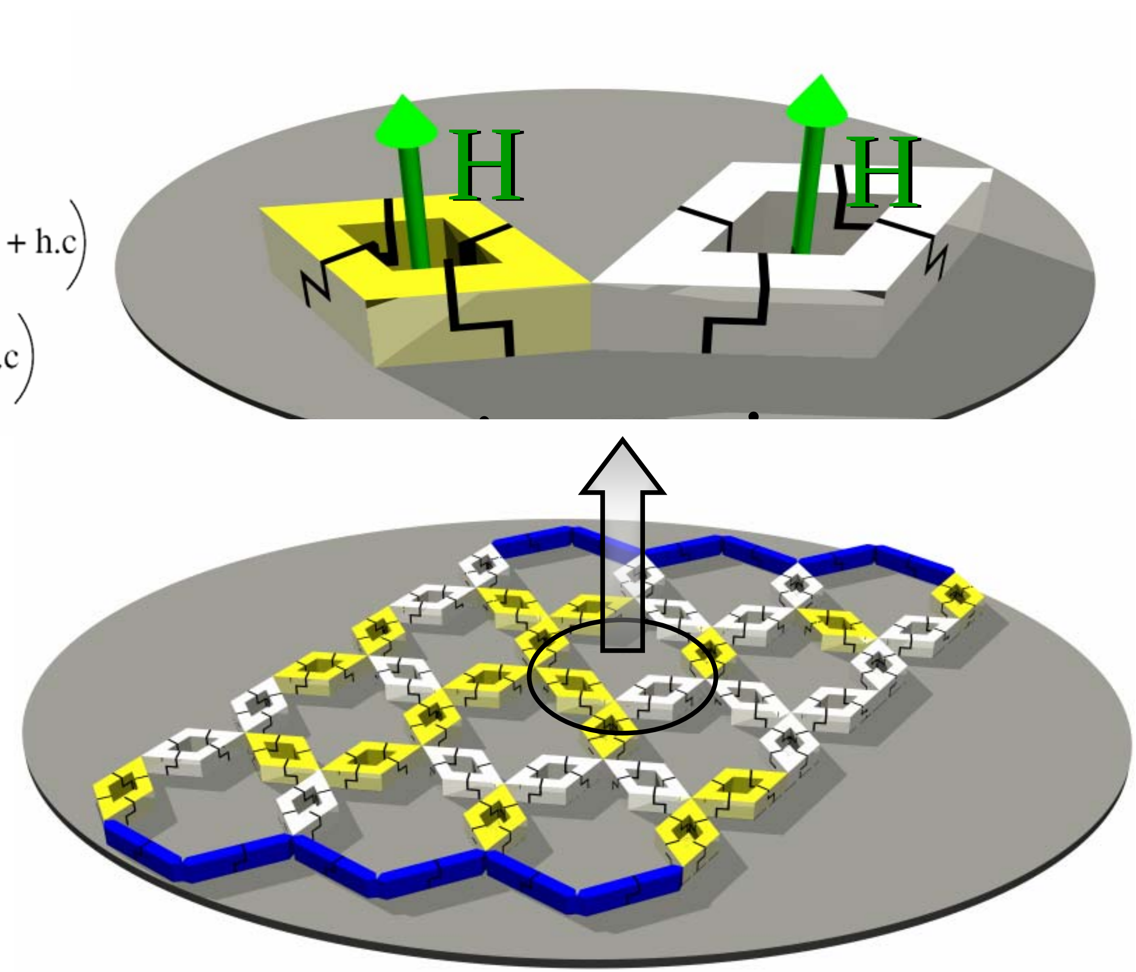


Questions

How can it be initialized, manipulated and read with *local* couplings ?

Topological qubits in Josephson junctions arrays : realization of RK model on a triangular lattice





Infinite protection against weak noise in thermodynamic limit

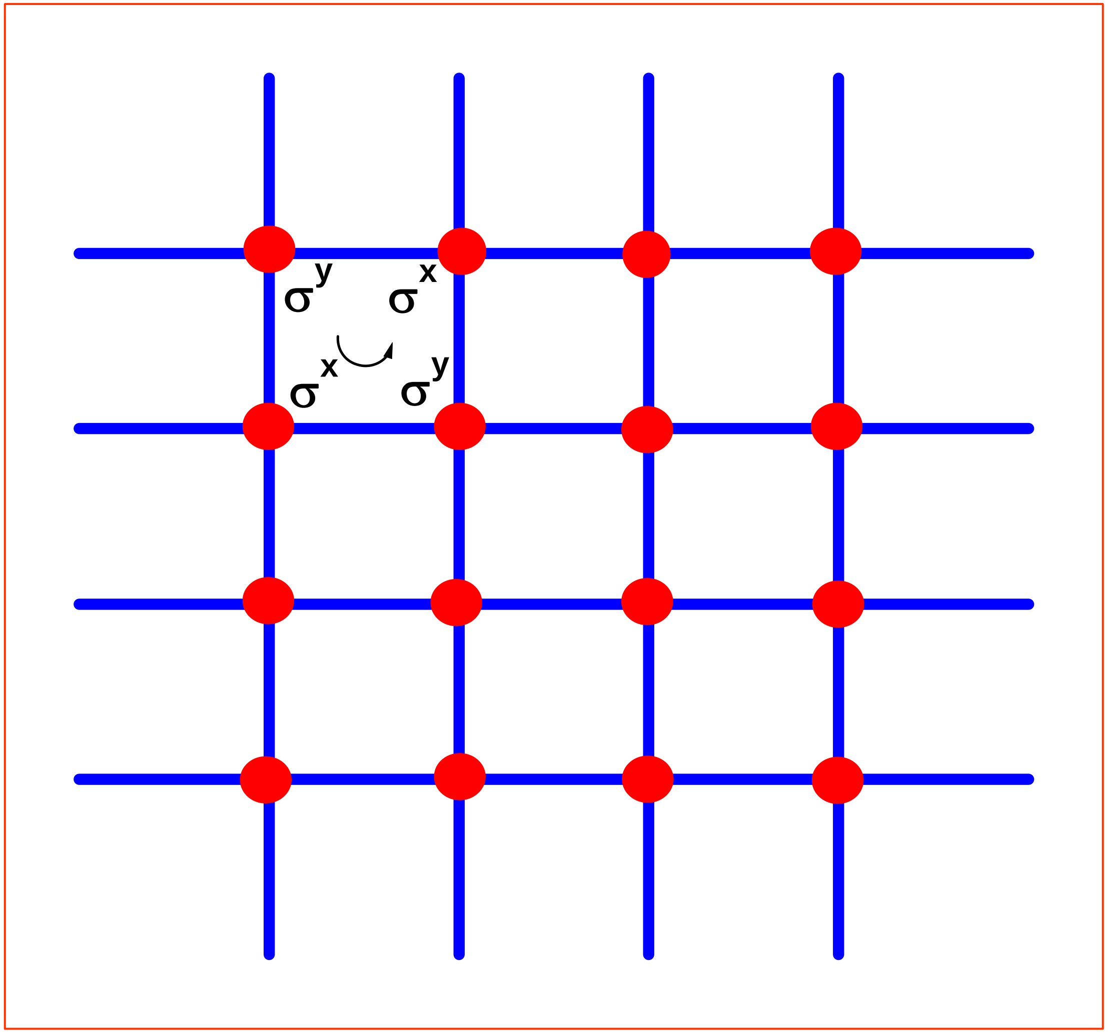
L. B. Ioffe, Nature 415, 503 (2002).

An exact solved model of Z2 topological order - Wen-plaquette model

$$H = -g\sum_i \hat{F}_i,$$

$$\hat{F}_i = \sigma_i^x \sigma_{i+\hat{e}_x}^y \sigma_{i+\hat{e}_x+\hat{e}_y}^x \sigma_{i+\hat{e}_y}^y$$

X. G. Wen, PRL. 90, 016803 (2003)

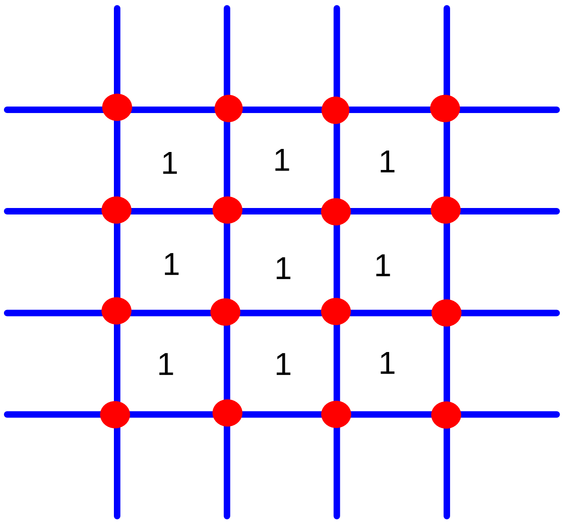


The energy gap

$$H = -g\sum_i \hat{F}_i, \quad \hat{F}_i = \sigma_i^x \sigma_{i+\hat{e}_x}^y \sigma_{i+\hat{e}_x+\hat{e}_y}^x \sigma_{i+\hat{e}_y}^y$$

- **For g>0, the ground state is**

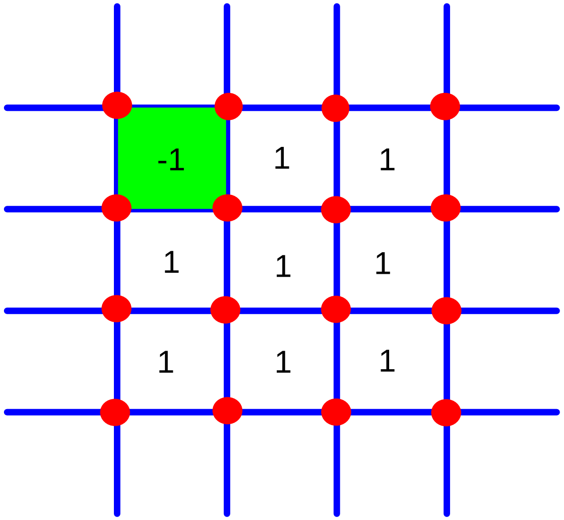
$$F_i = 1$$



The ground state energy is E_0=Ng

The elementary excitation is

$$F_i = -1$$



The energy gap for it becomes

$$E_1 - E_0 = 2g, \quad for \quad F_i = -1$$

The topological degeneracy on different lattices

- **The topological degeneracy depends on the lattice number $L_x \times L_y$: even-by-even, even-by-odd, odd-by-even, odd-by-odd.**

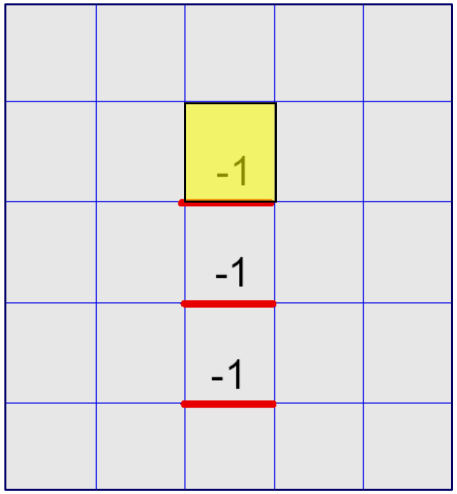
Even by even	Even by odd	Odd by even	Odd by odd
4	2	2	2

S. P. Kou, M. Levin, X. G. Wen, 0803.2300

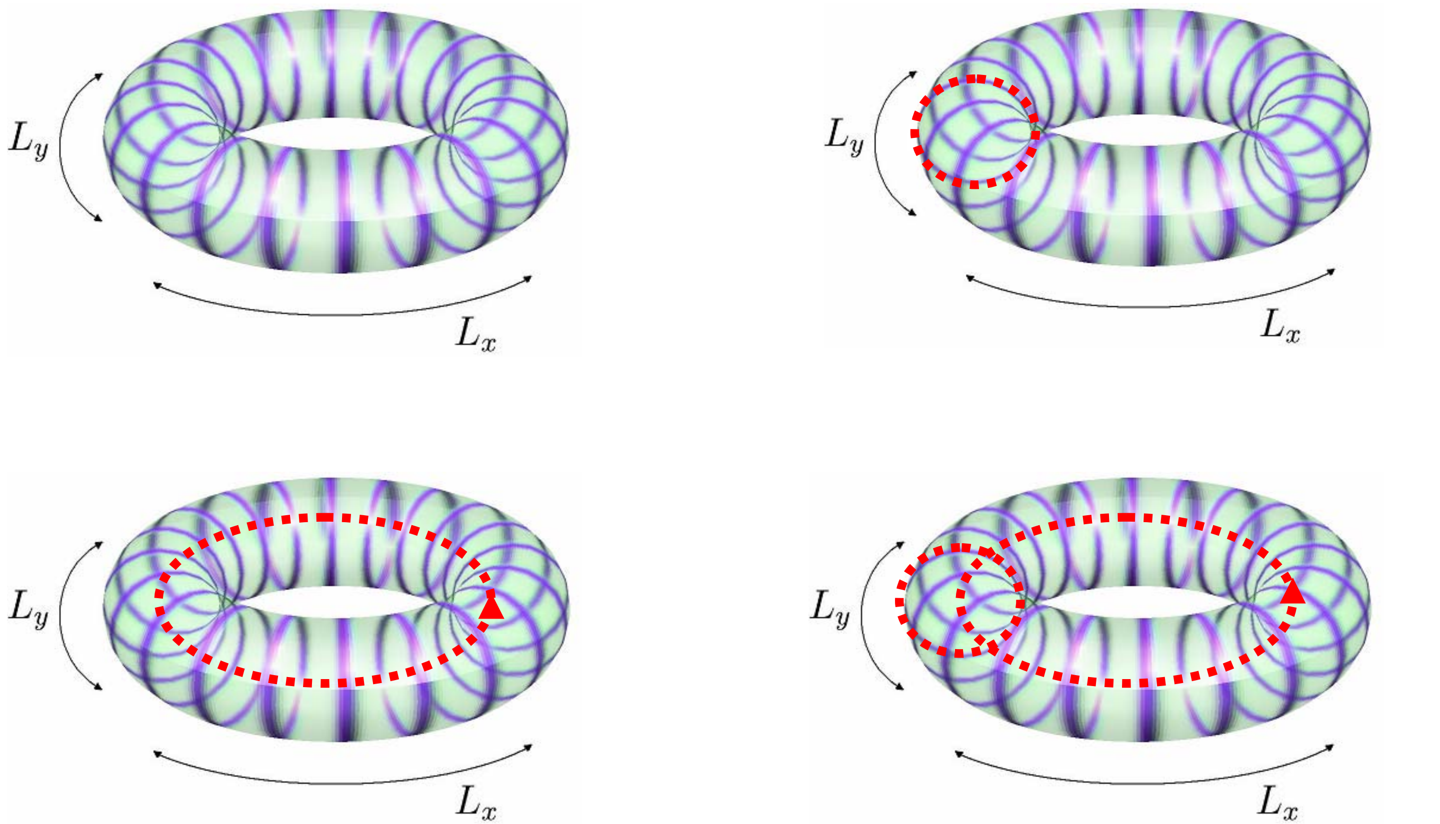
Topological characters of Z2 vortex (charge)

- **The existence of Z2 vortex (charge) :**

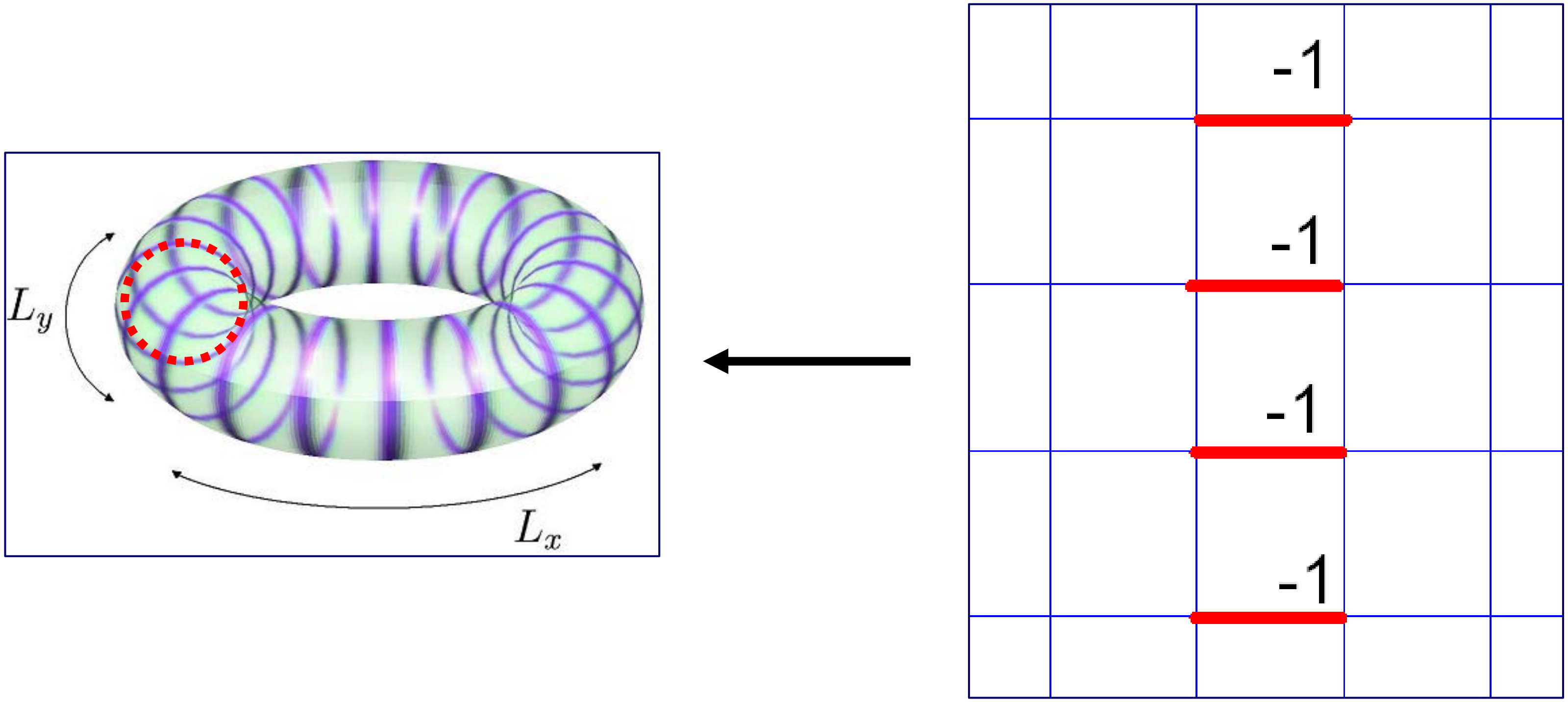
Z2 vortex (charge) is defined as a minus sigh of hopping term of fermions on one plaquette (or half line of minus sigh)

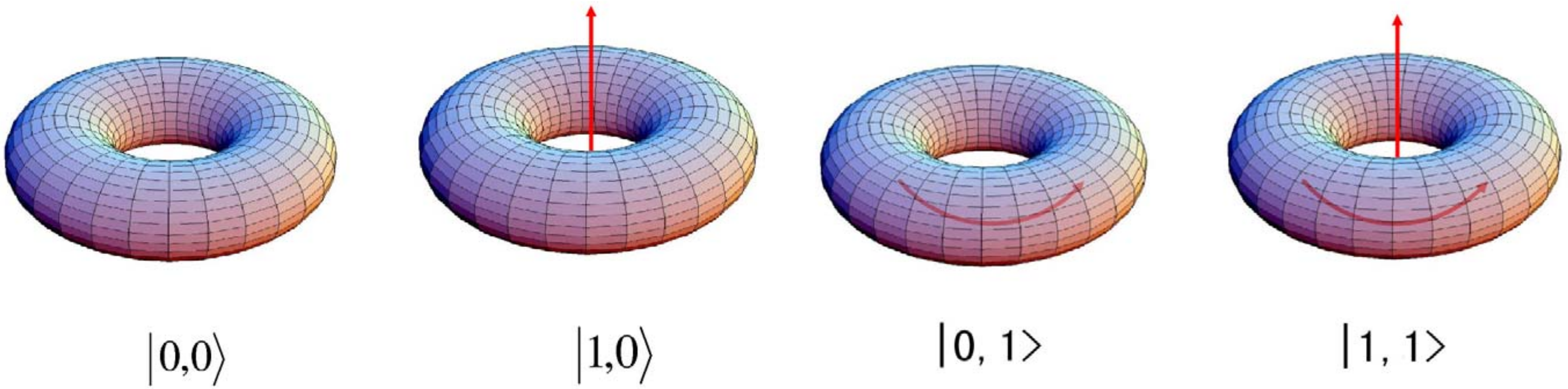


Ground states with 4-fold degeneracy



A line of negative sigh along y-direction is equal to the anti-periodic boundary condition along x-direction





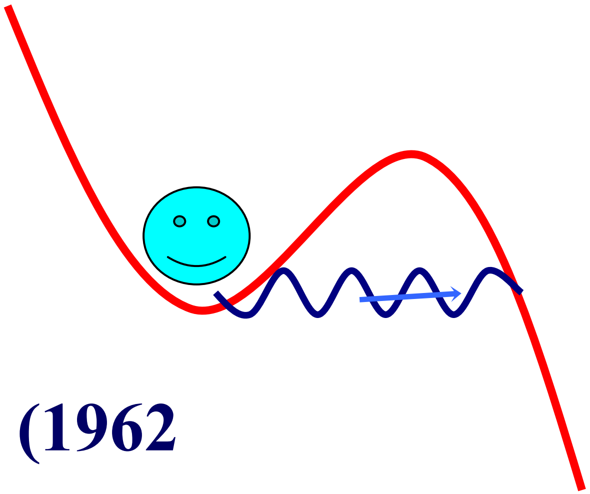
拓扑序的简并度是 **4** 指的是同一个能量对应四个不同的态，上图中用 **m, n = 0, 1** 来标记穿过环面上两个空洞的 π 通量的数目来表示不同的态。

$$\psi(x,\ y) = (-1)^m \psi(x,\ y+L_y),\ \psi(x,\ y) = (-1)^n \psi(x+L_x,\ y).$$

III. Quantum tunneling effect of Z2 topological order

1. Quantum tunneling effects

Gamow， Barrier penetration



- **Nuclear Energy**

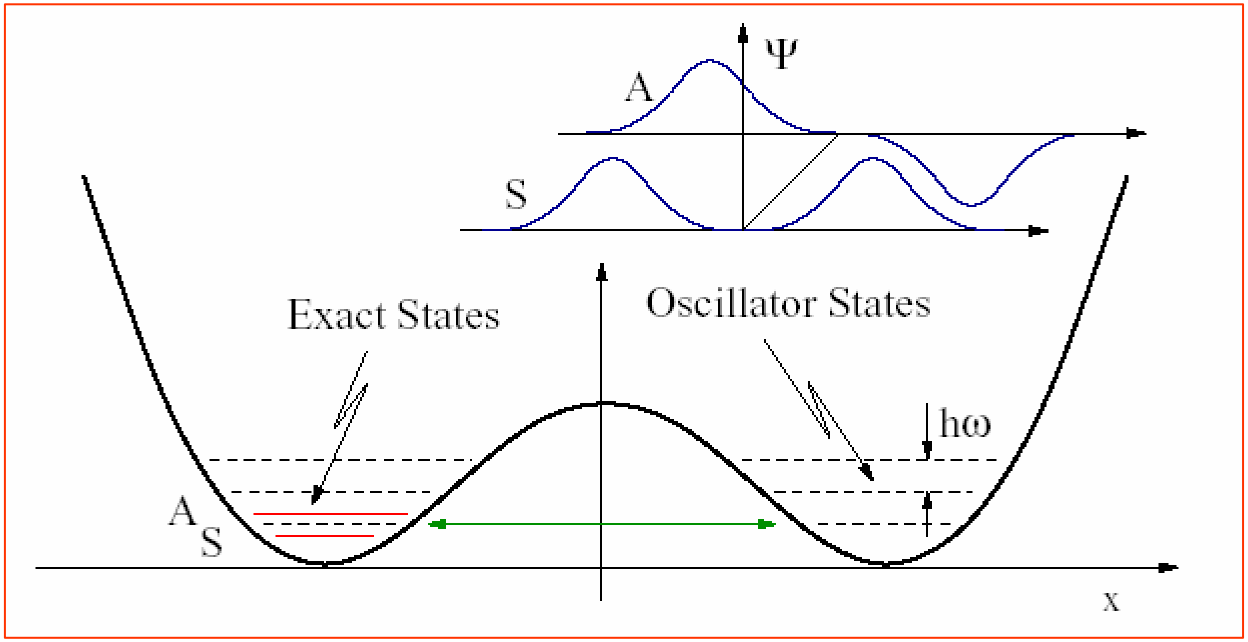
  Nuclear fusion – Hydrogen Bomb

- **Josephson junction – D. Josephson (1962**

- **Macroscopic quantum phenomena (MQP) of nano-magnetic particle**

- **…**

Quantum tunneling effect and level splitting of ground states in Double-Well



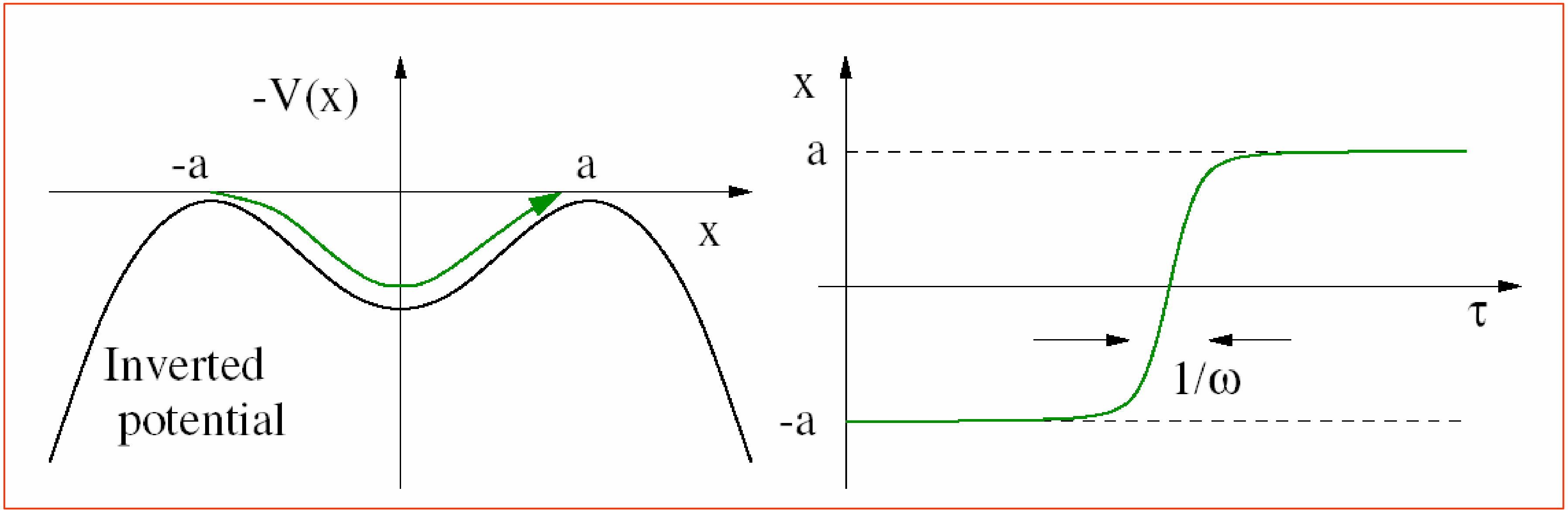
$$\Delta E = \frac{\hbar\omega}{\pi} e^{-S/\hbar}$$

$$S = \int_b^a \sqrt{2mV(x)}dx$$

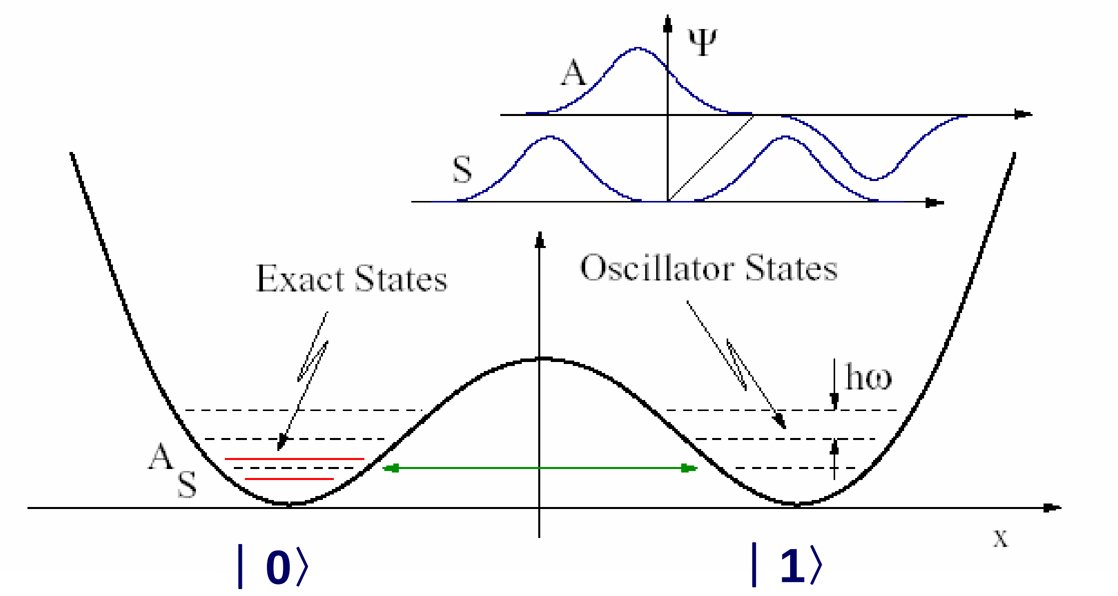
S : Euclidean action

ω : small oscillator frequency near the bottom of potential well

The action of instantons and quantum tunneling effects



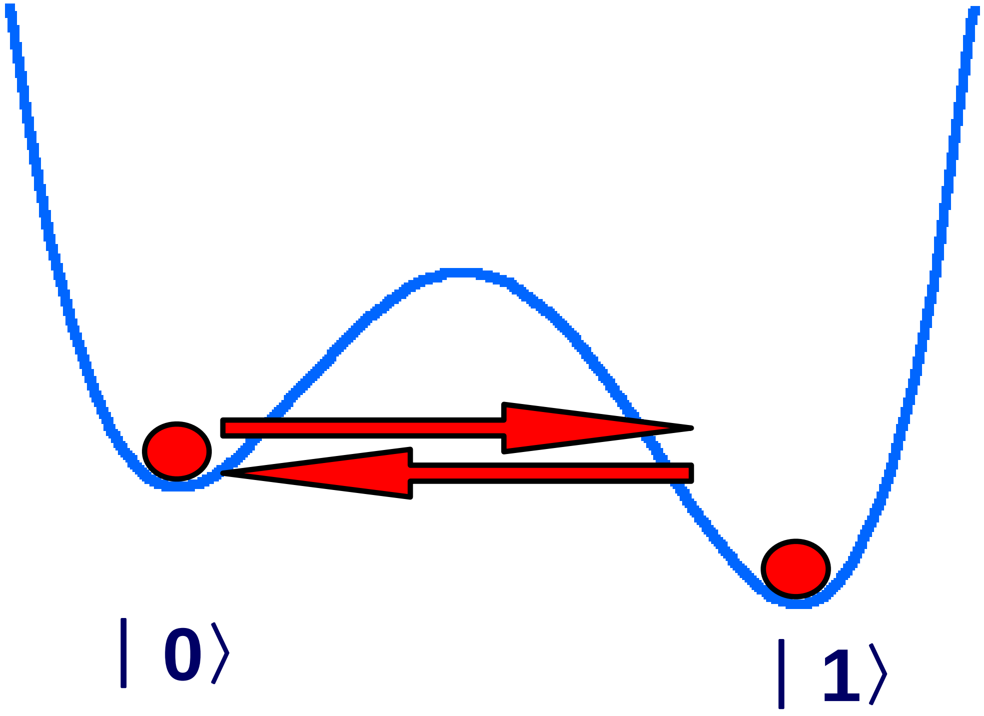
Pseudo-spin operator of quantum tunneling process



We use | 0⟩ and | 1⟩ to denote the ground states in two wells.

Quantum tunneling process | 0⟩ → | 1⟩ and | 1⟩ → | 0⟩ can be described by a pseudo-spin operator τ^x

Effective spin model of a general quantum tunneling process



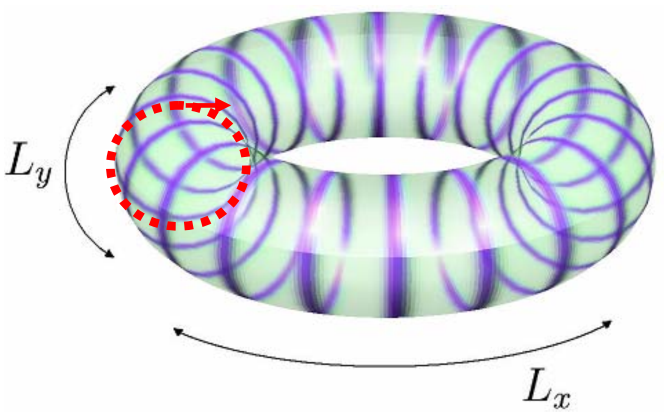
$$\begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix}$$

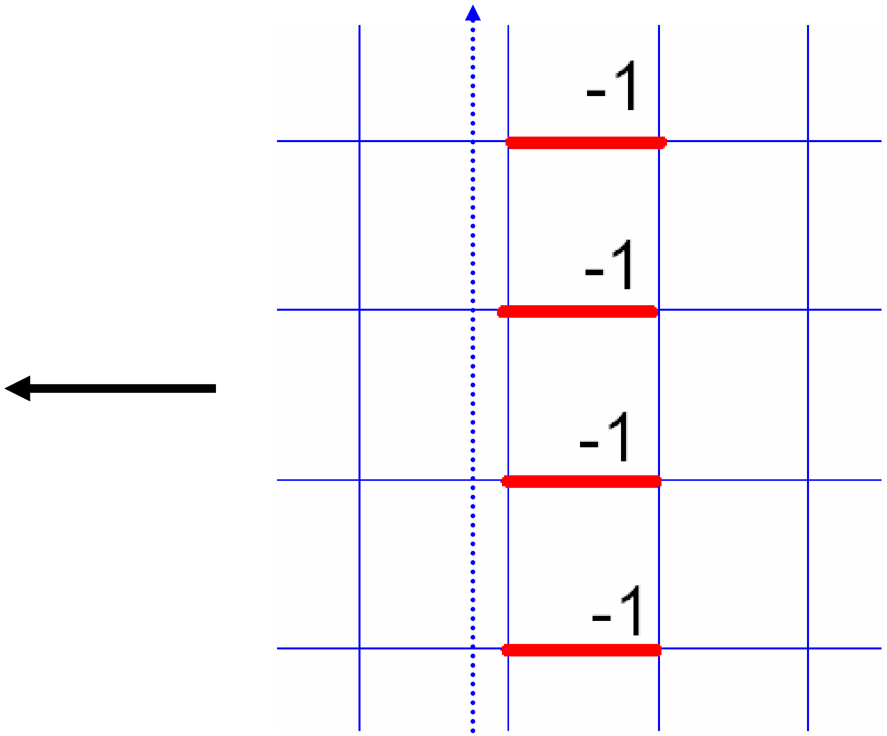
$$H_{eff} = a\tau^{x} + b\tau^{z}$$

$$b \propto \Delta E$$

2. Quantum tunneling effects in Z2 topological order

Tunneling processes : a virtual quasi-particle moves around the torus before annihilated with the other one.





Tunneling process of Z_2 vortex along x-direction

$|\,0,0\rangle \rightarrow |\,0,1\rangle$

$|\,1,0\rangle \rightarrow |\,1,1\rangle$

$|\,0,1\rangle \rightarrow |\,0,0\rangle$

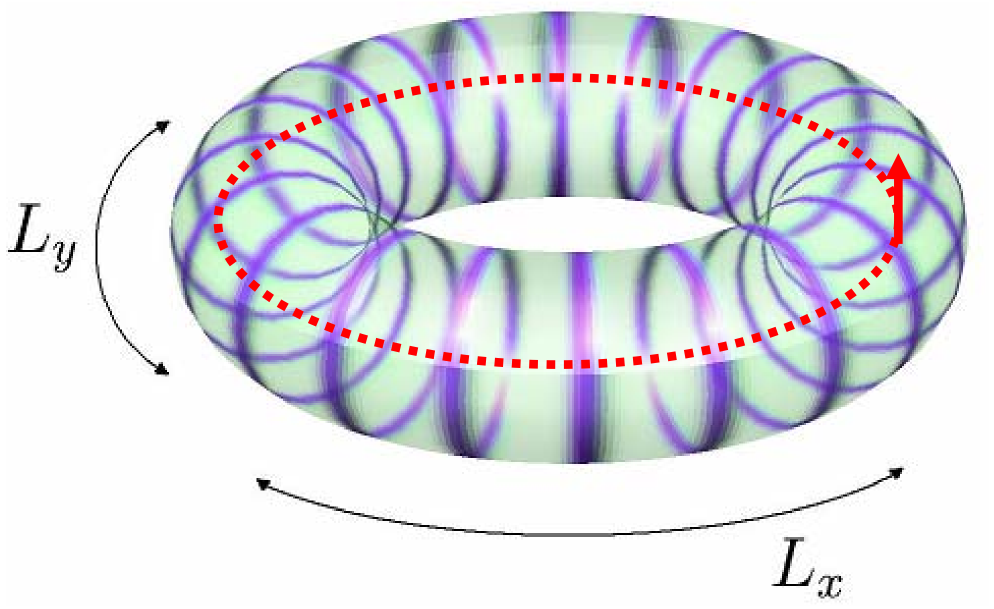
$|\,1,1\rangle \rightarrow |\,1,0\rangle$

pseudo – spin

operator

$\tau_1^x \otimes 1$

$$\begin{pmatrix} |00\rangle \\ |10\rangle \\ |01\rangle \\ |11\rangle \end{pmatrix}$$



Effective operators denote 9 tunneling processes

	Z2 vortex	Z2 charge	Fermion
x	$\tau_1^x \otimes 1$	$1 \otimes \tau_2^x$	$\tau_1^x \otimes \tau_2^x$
y	$\tau_1^x \otimes \tau_2^z$	$\tau_1^z \otimes \tau_2^x$	$\tau_1^y \otimes \tau_2^y$
x+y	$1 \otimes \tau_2^z$	$\tau_1^z \otimes 1$	$\tau_1^z \otimes \tau_2^z$

$$\begin{pmatrix} \lvert 00\rangle \\ \lvert 10\rangle \\ \lvert 01\rangle \\ \lvert 11\rangle \end{pmatrix}$$

S. P. Kou, arXiv:0805.2714

Calculate the ground state energy splitting from higher order (degenerate) perturbation approach

The instanton action can be derived by

$$\delta \mathrm{E}_{\mathrm{ij}}^{(\mathrm{s})} = \langle \varphi_i \,|\, \hat{H}'\left(\frac{\hat{H}'}{\hat{H}_0 - E_0}\right)^{s-1} |\, \varphi_j \rangle$$

$$S = L\,\mathbf{ln}\left(\frac{t}{E_0}\right)$$

S : Euclidean action

L : Hopping length of quasi-particles

t : Hopping integral

E_0: Excited energy of quasi-particles

Effective Hamiltonian

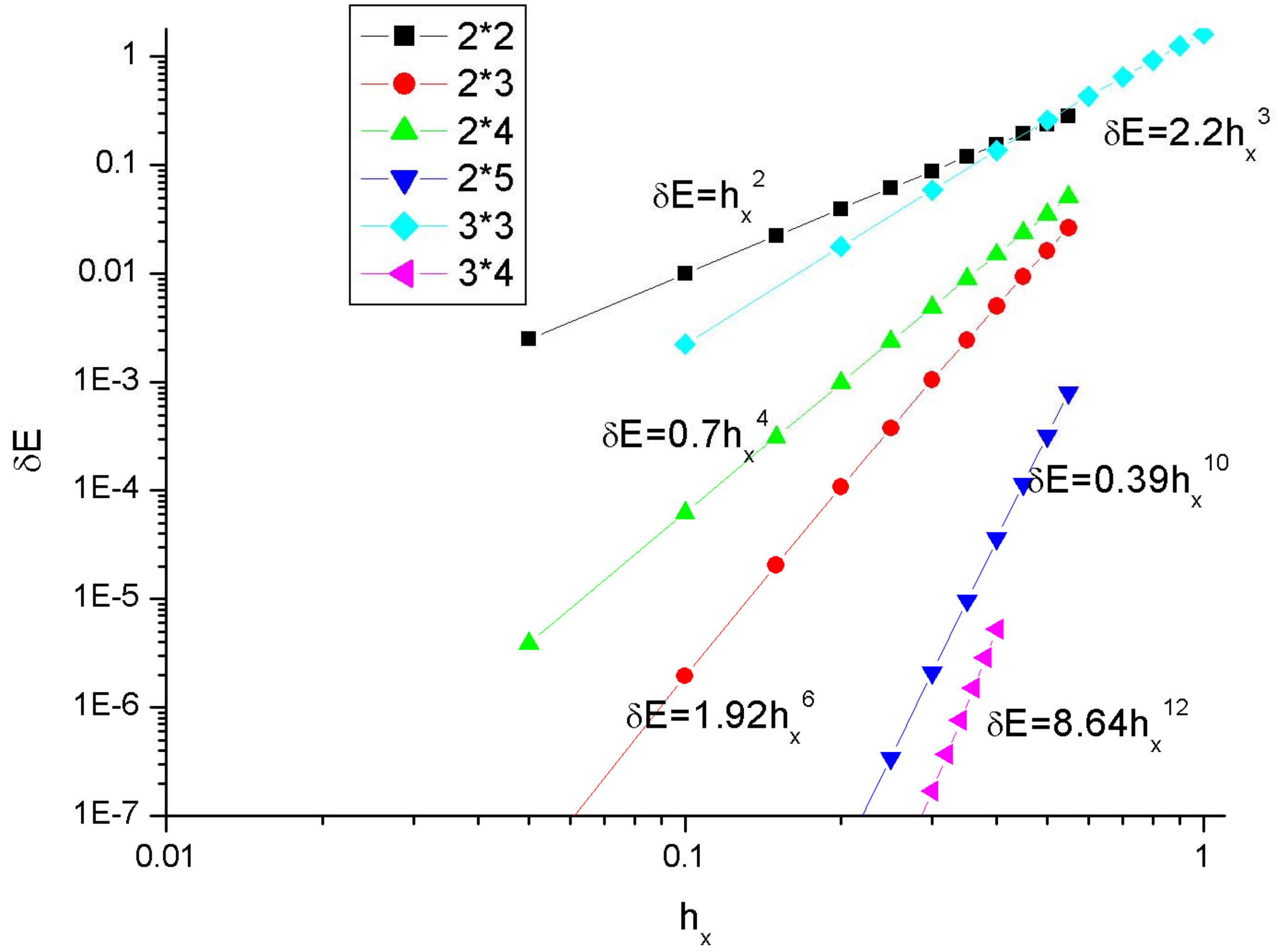
$$\mathcal{H}_{\rm eff} = J_{xx}\tau_1^x \cdot \tau_2^x + J_{yy}\tau_1^y \cdot \tau_2^y + J_{zz}\tau_1^z \cdot \tau_2^z + J_{zx}\tau_1^z \cdot \tau_2^x$$
$$+ J_{xz}\tau_1^x \cdot \tau_2^z + \tilde{h}_1^x \tau_1^x + \tilde{h}_1^z \tau_1^z + \tilde{h}_2^x \tau_2^x + \tilde{h}_2^z \tau_2^z$$

$$\mathrm{H}_{\rm eff} = \begin{pmatrix} J_{zz} + \tilde{h}_1^z + \tilde{h}_2^z & J_{zx} + \tilde{h}_2^x & J_{xz} + \tilde{h}_1^x & J_{xx} - J_{yy} \\ J_{zx} + \tilde{h}_2^x & -J_{zz} + \tilde{h}_1^z - \tilde{h}_2^z & J_{xx} + J_{yy} & -J_{xz} + \tilde{h}_1^x \\ J_{xz} + \tilde{h}_1^x & J_{xx} + J_{yy} & -J_{zz} - \tilde{h}_1^z + \tilde{h}_2^z & -J_{zx} + \tilde{h}_2^x \\ J_{xx} - J_{yy} & -J_{xz} + \tilde{h}_1^x & -J_{zx} + \tilde{h}_2^x & J_{zz} - \tilde{h}_1^z - \tilde{h}_2^z \end{pmatrix}$$

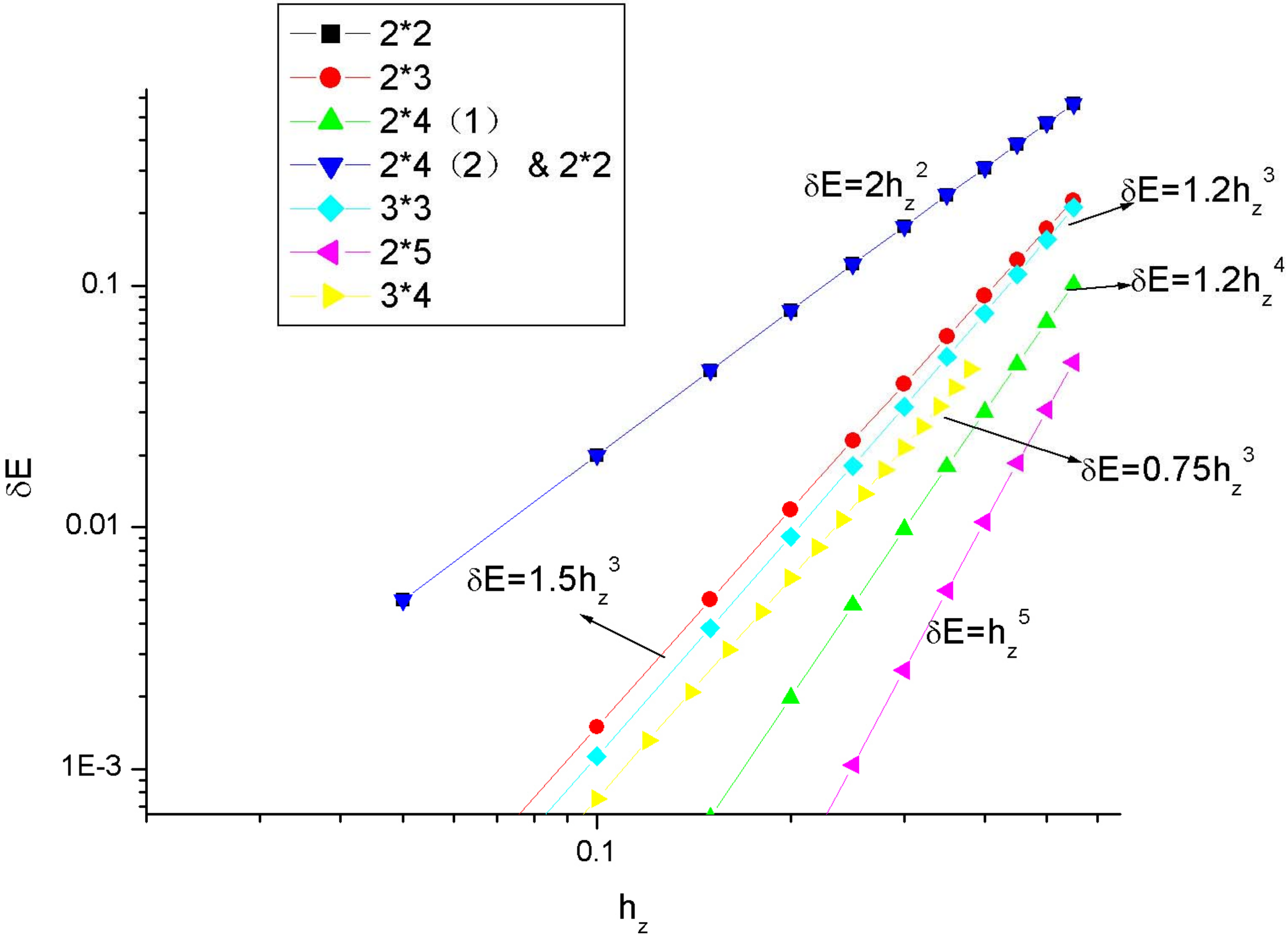
$\tilde{h}_1^x$、$\tilde{h}_2^x$、J_{xx}、$\tilde{h}_1^z$、$\tilde{h}_2^z$、J_{zz}、J_{xz}、J_{zx}、J_{yy}

S. P. Kou, arXiv:0805.2714

Ground states energy splitting of Wen-plaqutte model under a magnetic field along x-direction

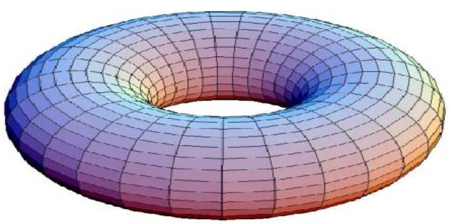


Ground states energy splitting of Wen-plaqutte model under a magnetic field along z-direction

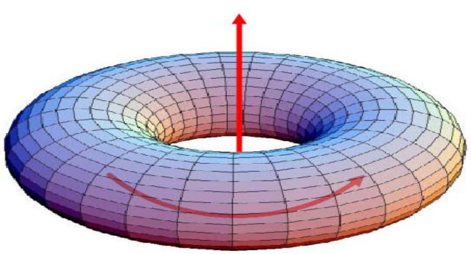


IV. Topological quantum computation by manipulating quantum tunneling Effect

1. **Qubit :**
   degenerate ground states of Wen-plaquette model on odd-by-odd lattice



$|0,0\rangle$



$|1,1\rangle$

How to initial, do operation, measure?

2. Initialization

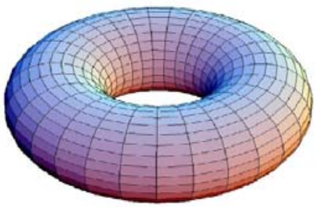
- **Applied a external fields along z-directions, and then reduce it to zero :**

$$H' = h(t) \sum_i \sigma_z \quad \text{where} \quad h(t) = e^{-t/t_0} - 1$$

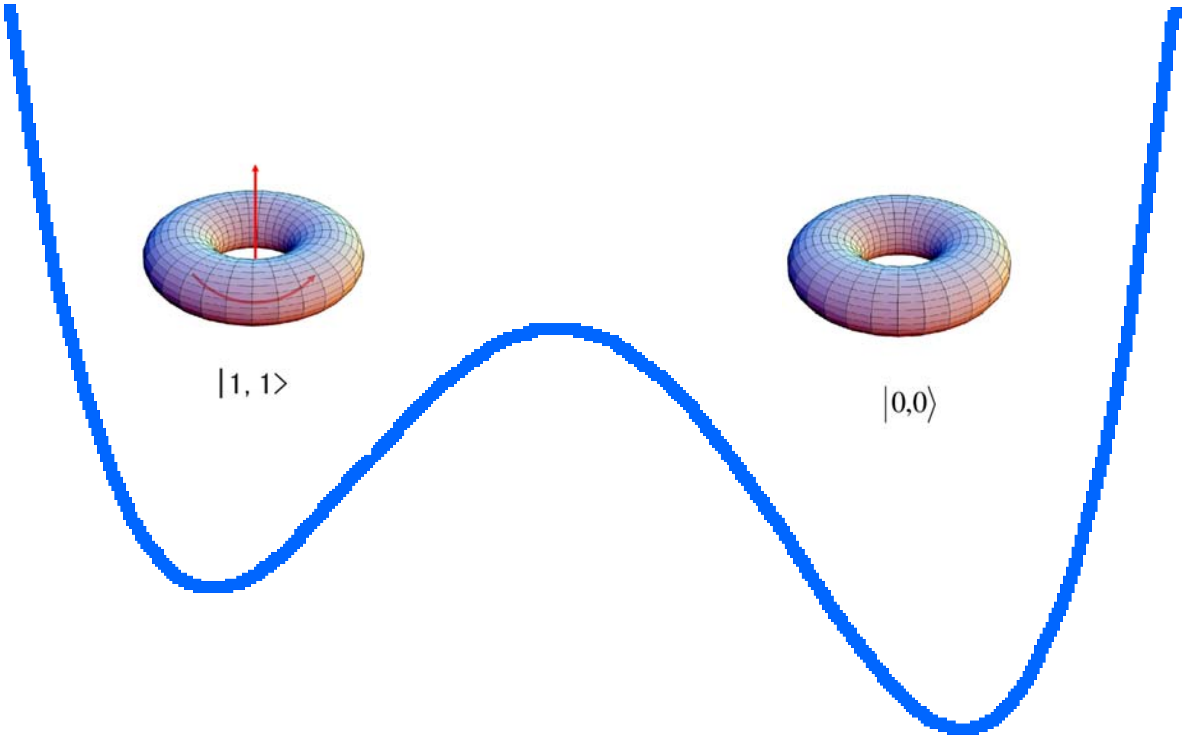
- **The unitary operator becomes :**

$$U(t) = e^{\frac{-iH't}{\hbar}}$$

- **Finally at t=0, we have the state $|0,0\rangle$**



$|0,0\rangle$


|1, 1⟩
|0,0⟩

3. Unitary operations

- **A general operator becomes :**

$$U_{\theta,\varphi} = e^{-\frac{i}{\hbar}\gamma\tau^z} e^{-\frac{i}{\hbar}\varphi(\tau^x+\tau^y)} e^{-\frac{i}{\hbar}\theta\tau^z}$$

where $\gamma = J_{zz}\Delta t_\gamma$, $\theta = J_{zz}\Delta t_\theta$ and $\varphi = J\Delta t_\varphi$

For example , Hadamard gate is

$$U_{\theta,\varphi}(\gamma = \frac{\pi}{4}, \theta = \frac{7\pi}{4}, \varphi = \frac{\pi}{4})$$

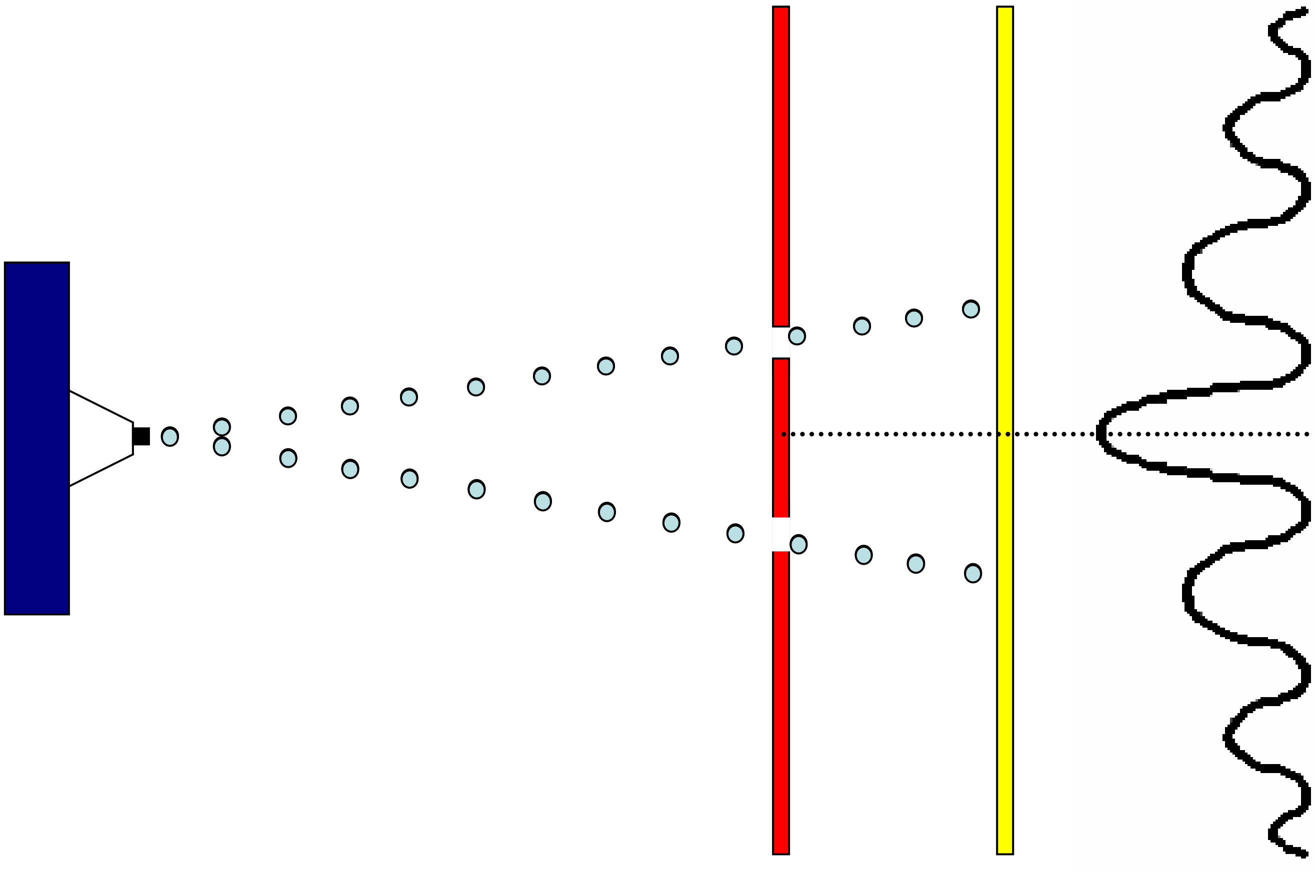
4. Measurement

- **We want to determine the state**

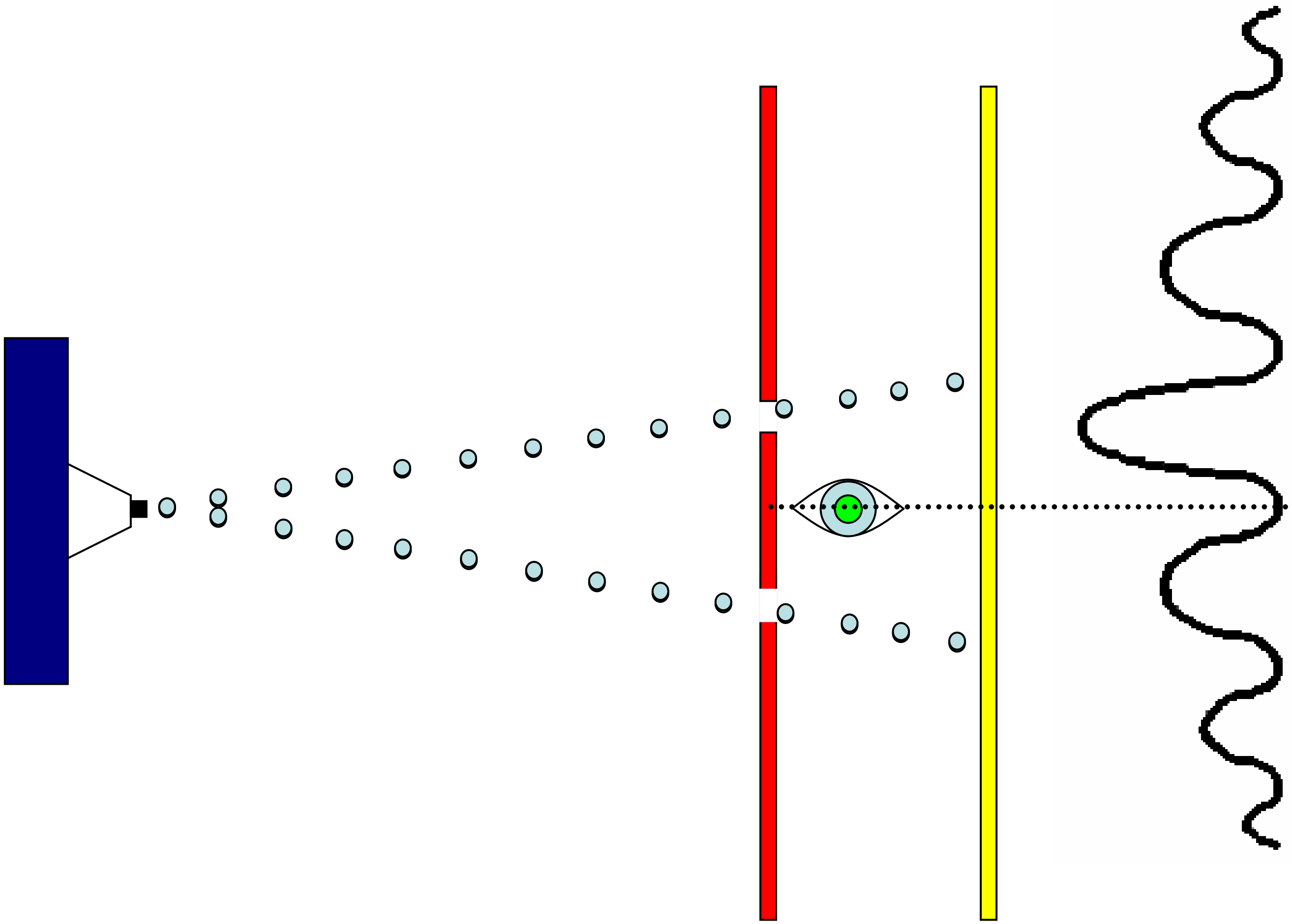
$$\left|vac\right\rangle = \alpha\left|00\right\rangle + \beta e^{i\phi}\left|11\right\rangle$$

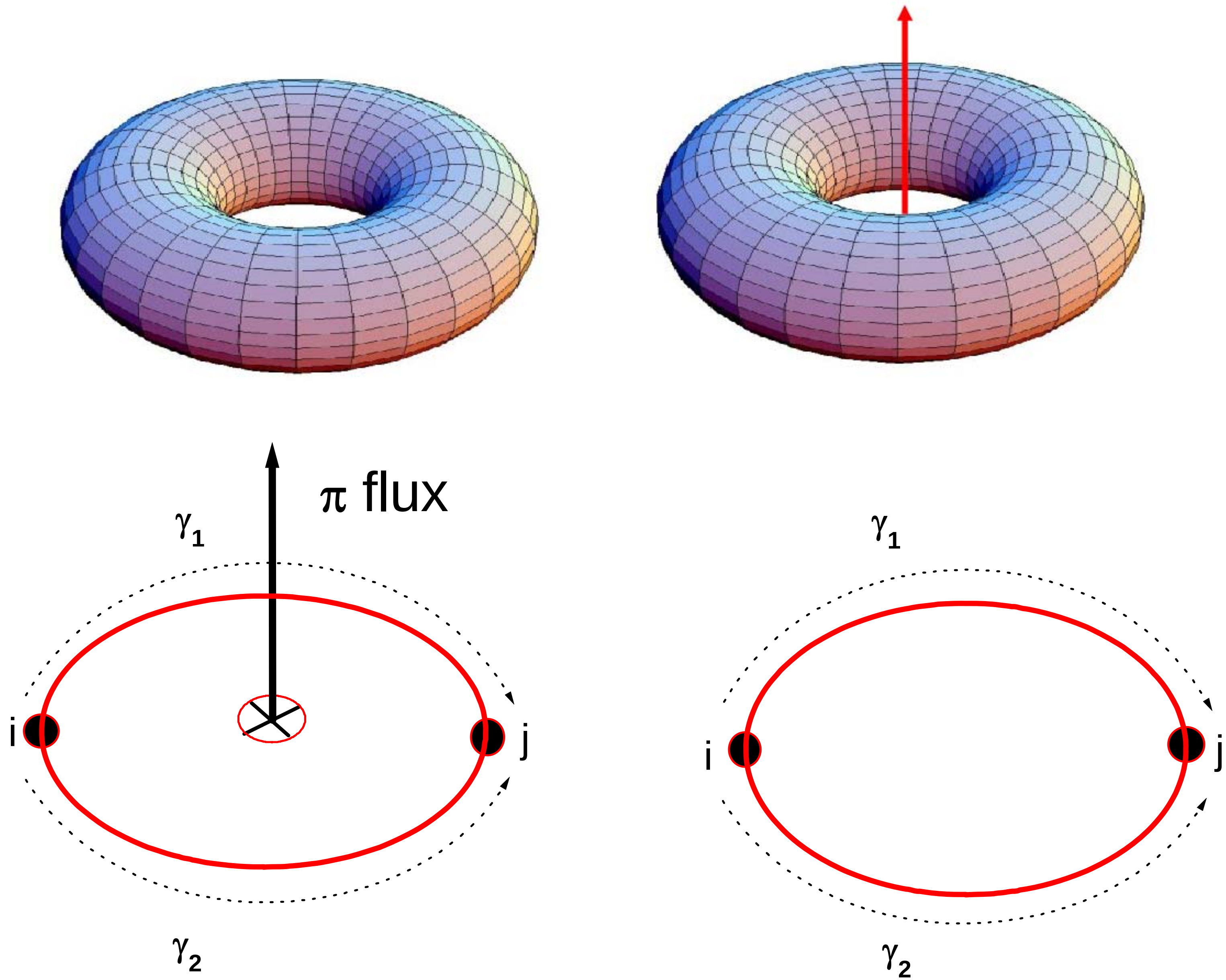
- **The interference from Aharonov-Bohm (AB) effect allows one to observe distinction between the processes with or without a flux inside the loop.**

Interference in double slits



Observing AB effect in double slits





π flux
γ1
γ2
i
j
γ1
γ2
i
j

5. Unsolved problems

- **It is a challenge to realize the designed spin model on a manifold of higher genus in the optical lattice of cold atoms.**

- **The coefficient of the energy splitting of degenerate ground states cannot calculated exactly.**


Thank You!

1. Errors

- Thermal effect : at finite temperature, anyons exist, their braiding leads to error.

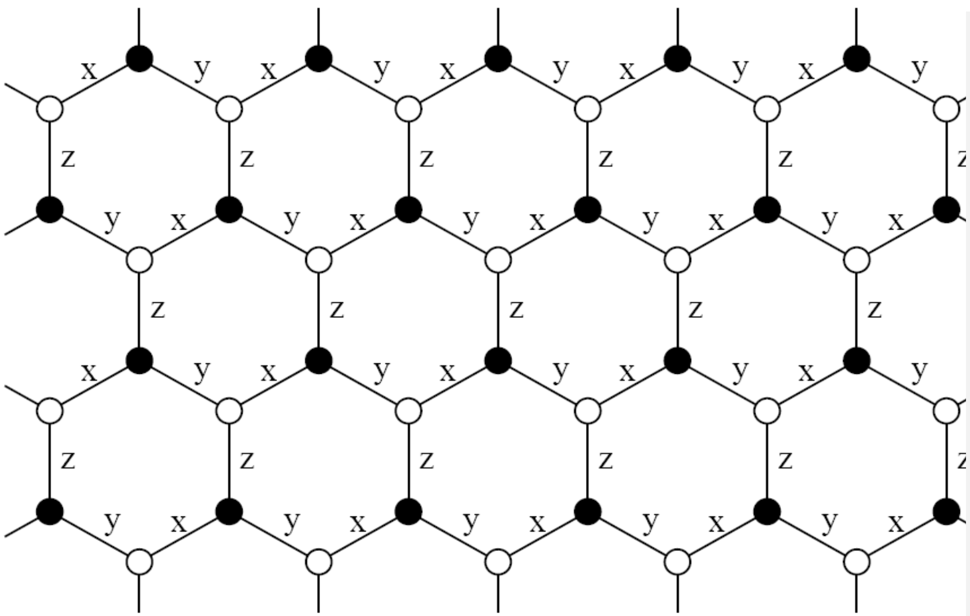
- One can not control the operator time exactly.

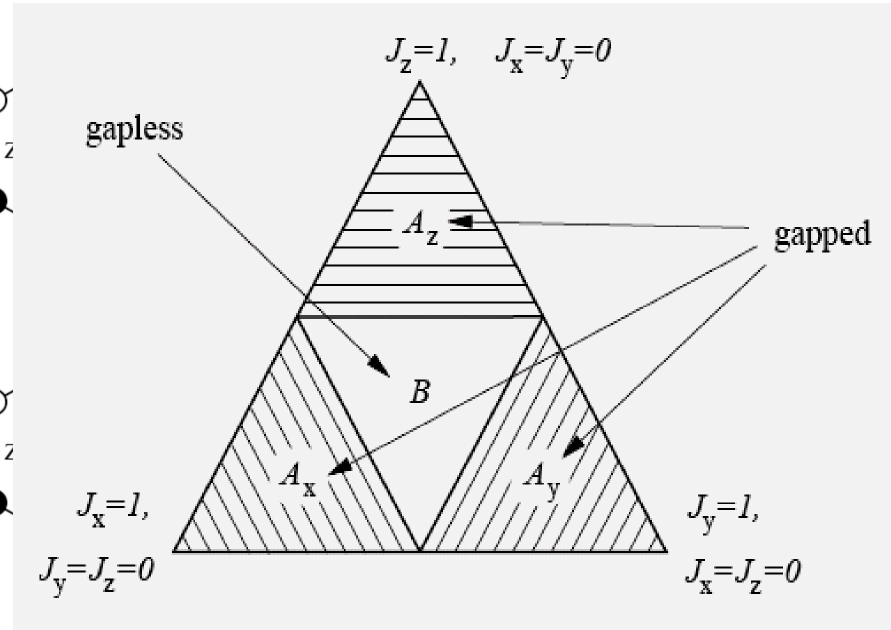
- When measurement, the ground state may still evolves.

2. Possible realization in cold atoms

Kitaev Model

$$H = J_1 \sum_{x-link} \sigma_n^x \sigma_m^x + J_2 \sum_{y-link} \sigma_n^y \sigma_m^y + J_3 \sum_{z-link} \sigma_n^z \sigma_m^z$$



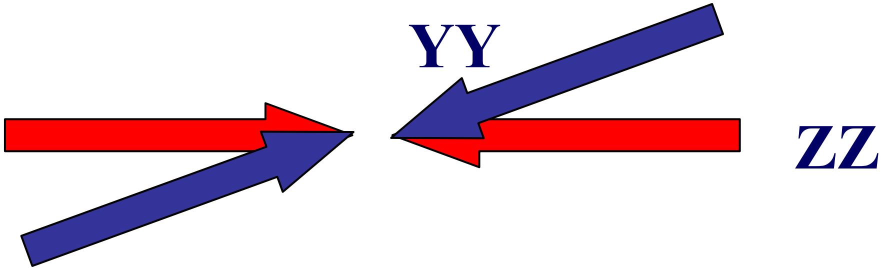


The Abel gapped phases A_x, A_y, A_z are Z2 topological orders

A. Kitaev, Ann Phys 321, 2 (2006)

Engineering the toy model

• Optical lattice in 2 dimensions: polarizations & frequencies of standing waves can be different for different directions

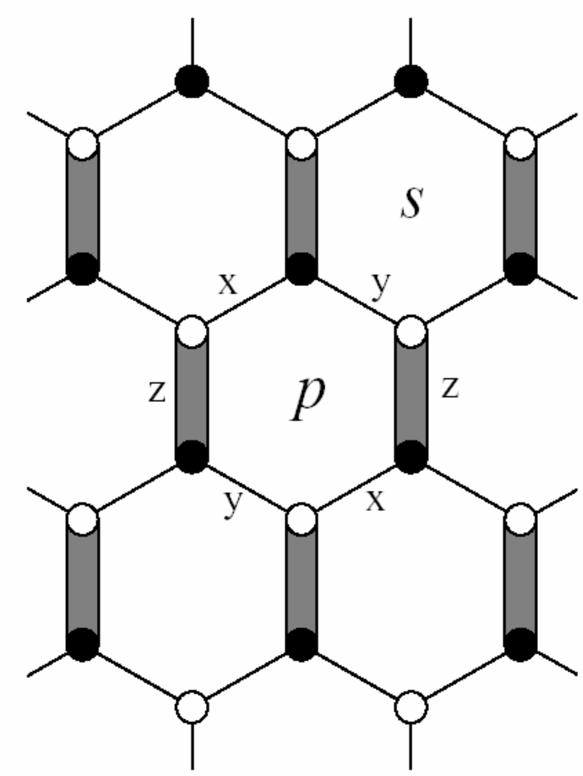


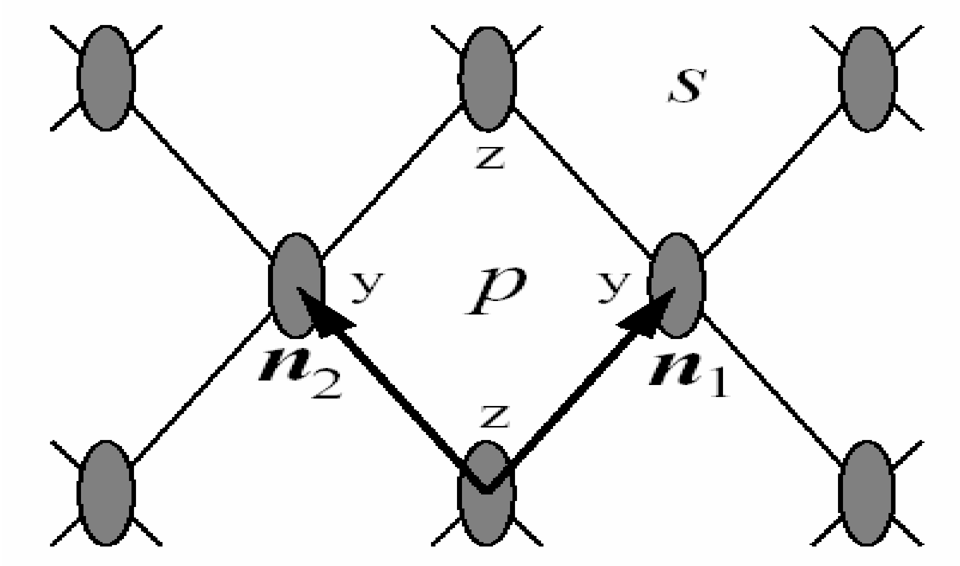
Kitaev model on honeycomb lattice Can be created with 3 sets of standing wave light beams.

L.-M. Duan, E. Demler, and M. D. Lukin, Phys. Rev. Lett. 91, 090402 (2003).



Wen-plaquette model as low energy effective model of Kitaev model on honeycomb lattice





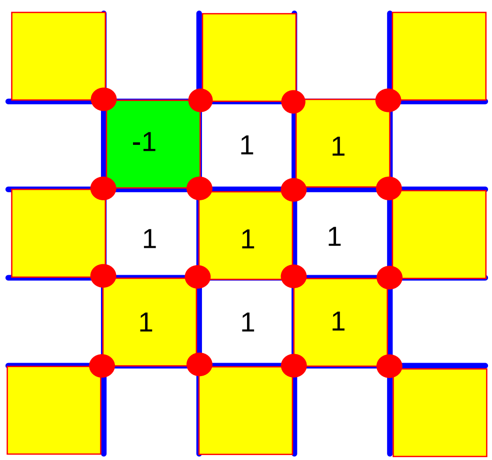
$$H = J_1 \sum_{x-link} \sigma_n^x \sigma_m^x + J_2 \sum_{y-link} \sigma_n^y \sigma_m^y + J_3 \sum_{z-link} \sigma_n^z \sigma_m^z$$

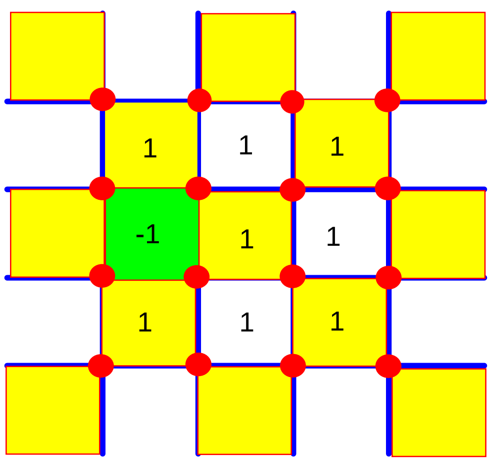
$$H_{\text{eff}} = -\frac{J_x^2 J_y^2}{16|J_z|^3} \sum_p Q_p$$

$$Q_p = \sigma^y_{\text{left}(p)} \sigma^y_{\text{right}(p)} \sigma^z_{\text{up}(p)} \sigma^z_{\text{down}(p)}$$

3. The statistics for the elementary excitations

- There are two kinds of Bosonic excitations:
- Z2 vortex $$F_{i \in i_x + i_y = even} = -1$$
- Z2 charge $$F_{i \in i_x + i_y = odd} = -1$$
- Each kind of excitations moves on each sub-plaquette:
- Why?



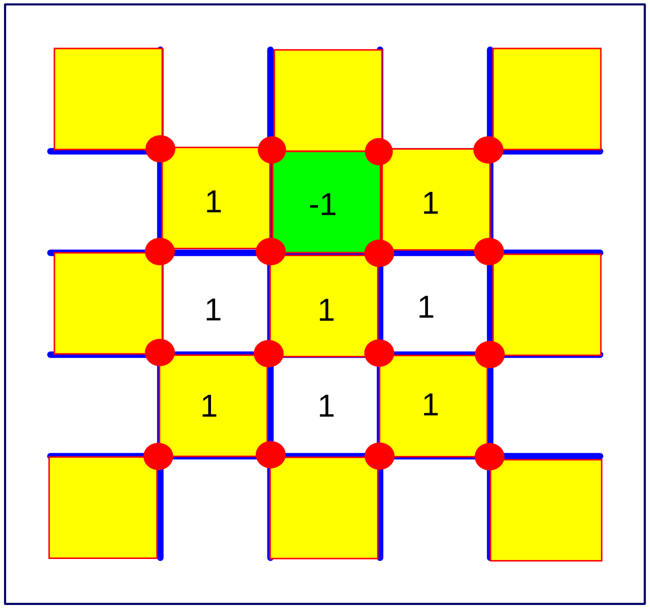


- **There are two constraints (the even-by-even lattice): One for the even plaquettes, the other for the odd plaquettes**

$$\prod_{i \in i_x + i_y = even} F_{i \in i_x + i_y = even} = 1 \qquad \prod_{i \in i_x + i_y = odd} F_{i \in i_x + i_y = odd} = 1$$

- **The hopping from even plaquette to odd violates the constraints**

- **The Z2 vortex turns into Z2 charge**



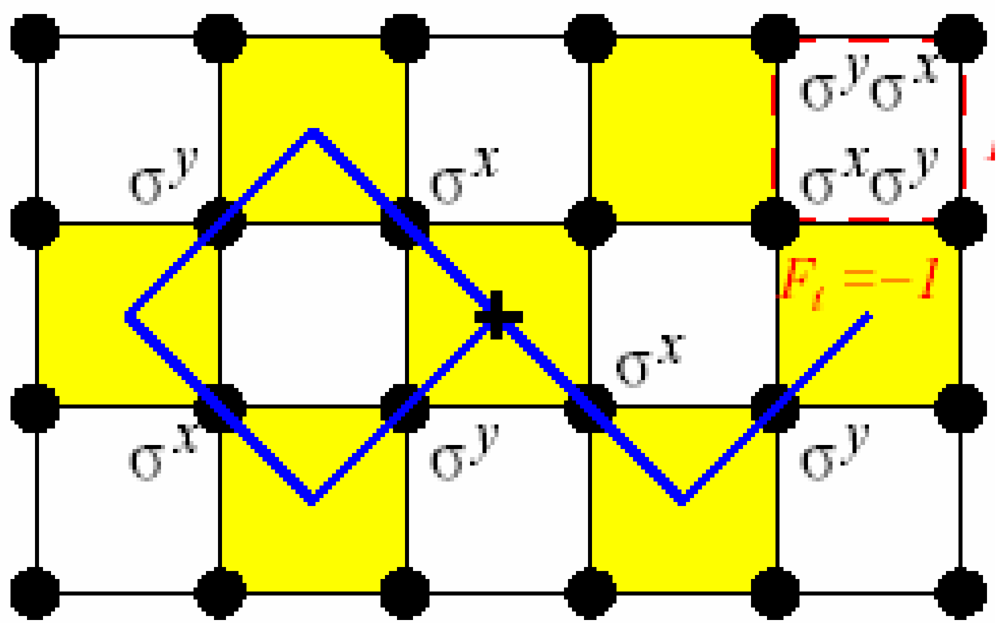
The dynamics of the Z2 Vortex and Z2 charge

- **Z2 vortex (charge) can only move in the same sub-plaquette:**
- **The hopping operators for Z2 vortex (charge) are**

$$\sigma_i^x \quad and \quad \sigma_i^y$$

$$\sigma^x{}_{i+\hat{e}_y}\hat{F}_i\sigma^x{}_{i+\hat{e}_y} = -\hat{F}_i$$

$$\sigma^y{}_i\hat{F}_i\sigma^y{}_i = -\hat{F}_i$$



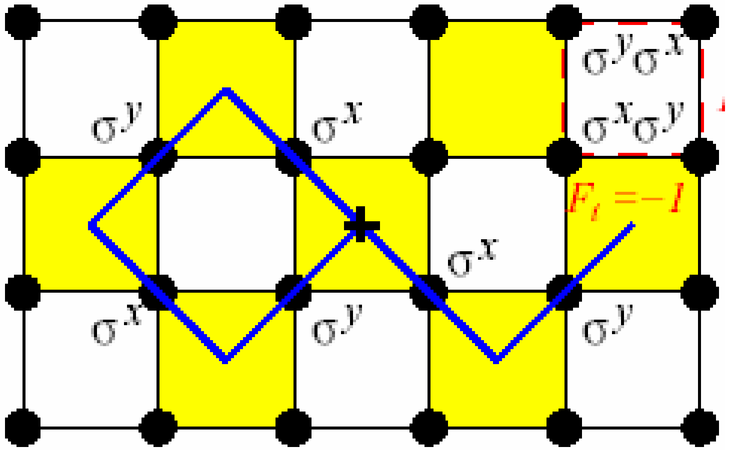
The mutual semion statistics between the Z2 Vortex and Z2 charge

- **When an excitation (Z2 vortex) in even-plaquette move around an excitation (Z2 charge) in odd-plaquette, the operator is** $F_i = \sigma_i^x \sigma_{i+\hat{e}_x}^y \sigma_{i+\hat{e}_x+\hat{e}_y}^x \sigma_{i+\hat{e}_y}^y$

- **it is -1 with an excitation on it**

$$F_i = -1$$

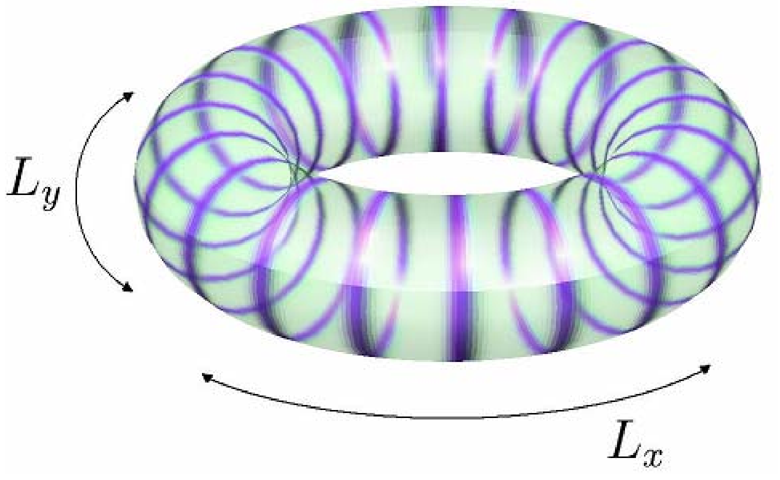
- **This is the character for semion statistics**



X. G. Wen, PRD**68**, 024501 (2003).

Topological degeneracy on a torus (even-by-even lattice) :

- On an even-by-even lattice, there are totally 2^N states
- Under the constaint, $\prod_{i_x+i_y=even} F_i^2 = 1$ *and* $\prod_{i_x+i_y=odd} F_i^2 = 1$ the number of states are only $2^N/4$
- For the ground state $F_i = 1$, it must be four-fold degeneracy.

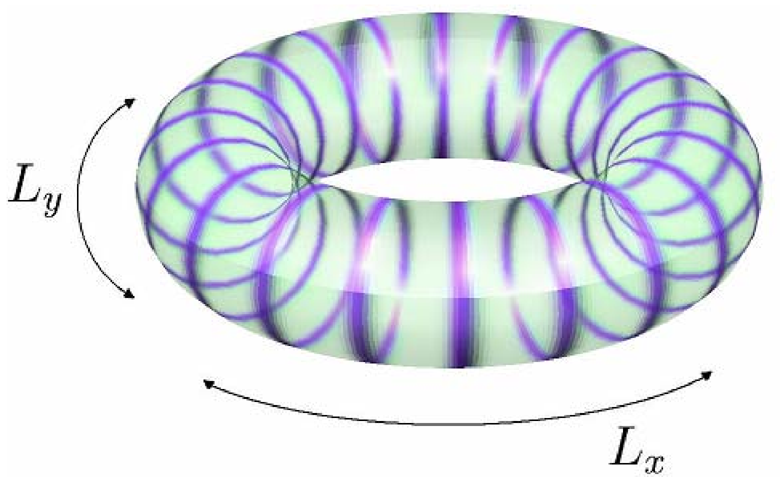


Topological degeneracy on a torus: even-by-odd, odd-by-even, odd-by-odd lattices

- There is only one constaint, there are only

$$\prod_i F_i^2 = 1$$

- For the ground state $F_i = 1$, it is two-fold degeneracy.

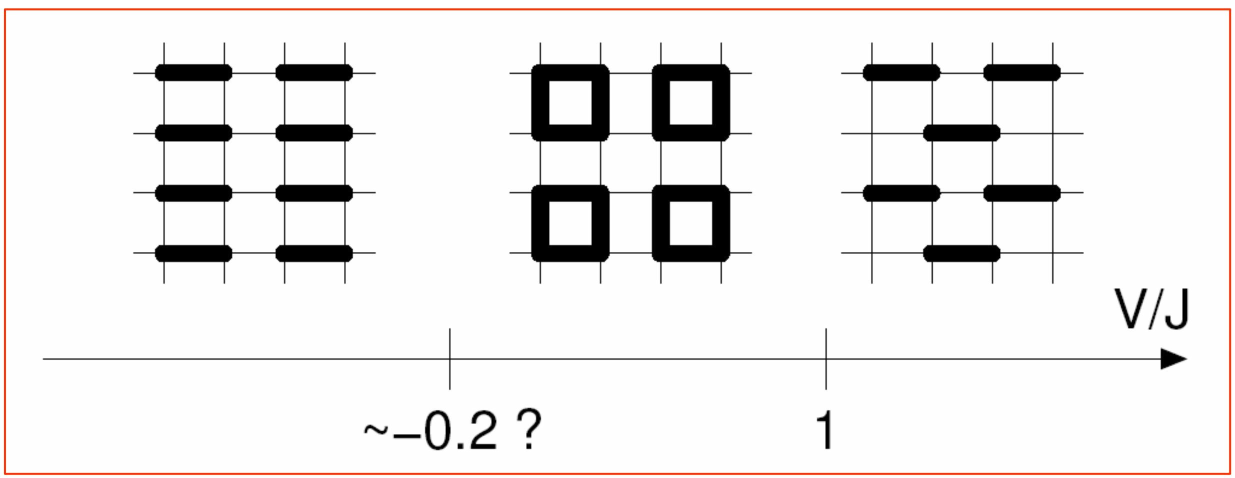


4. First example of Z2 topological order - Quantum Dimer Models

Square lattice (Rokhsar-Kivelson, '88)

$$\mathcal{H} = \sum_{\mathrm{Plaquette}} \left[-J \left(|\,\Vert\,\rangle \langle\,=\,| + \mathrm{H.c.} \right) + V \left(|\,\Vert\,\rangle \langle\,\Vert\,| + |\,=\,\rangle \langle\,=\,| \right) \right]$$

Assume dimer configurations are orthogonal

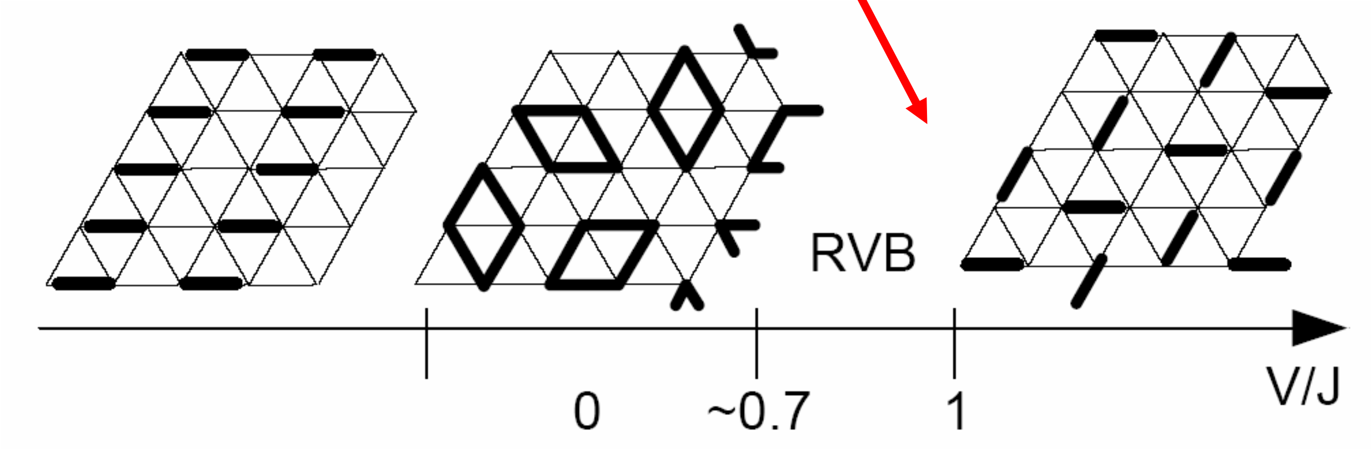


RK '88
Leung et al, '96

QDM on triangular lattice

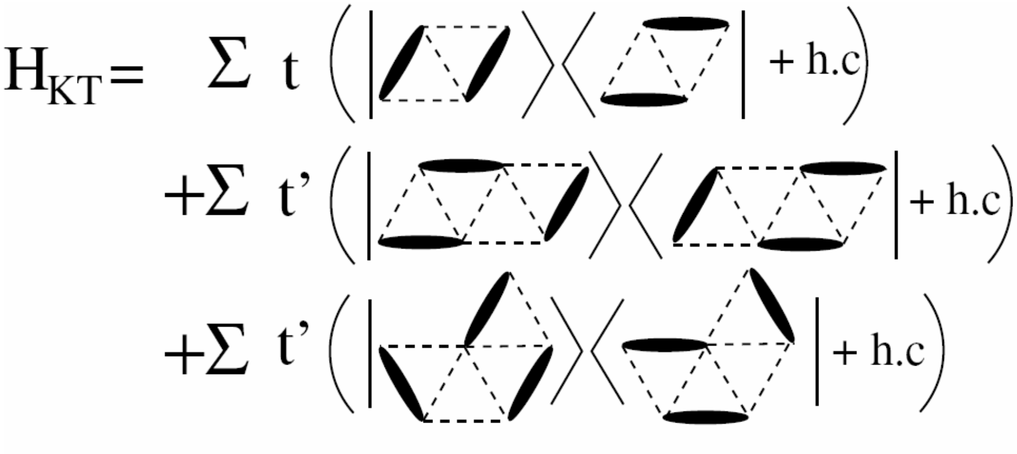
Moessner and Sondhi, '01

RVB spin liquid with gapped spectrum



Topological degeneracy

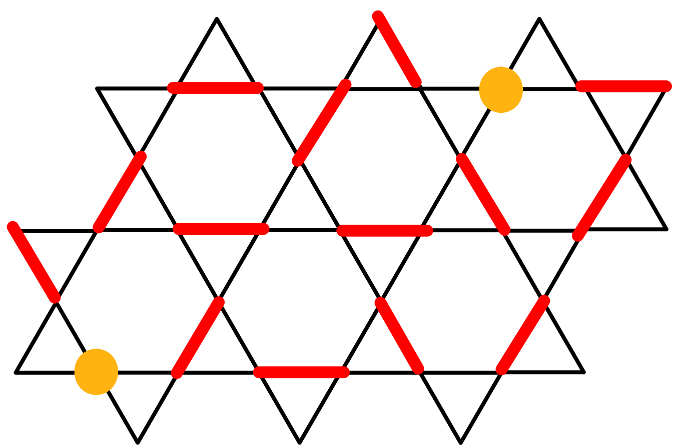
Rokhsar-Kivelson (RK) model on a triangular lattice

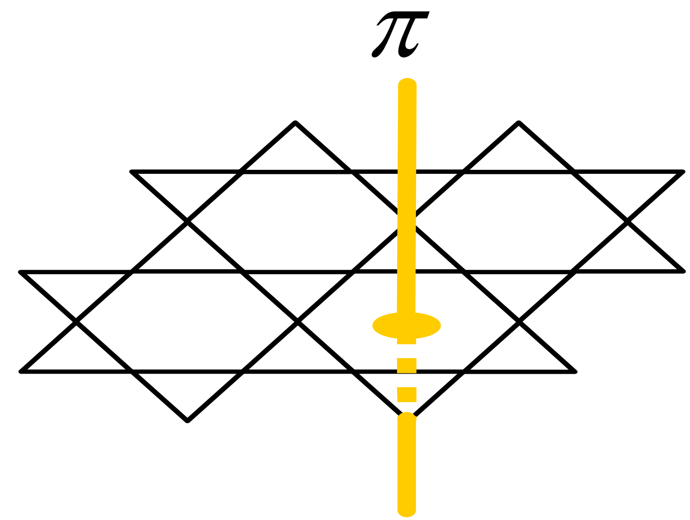


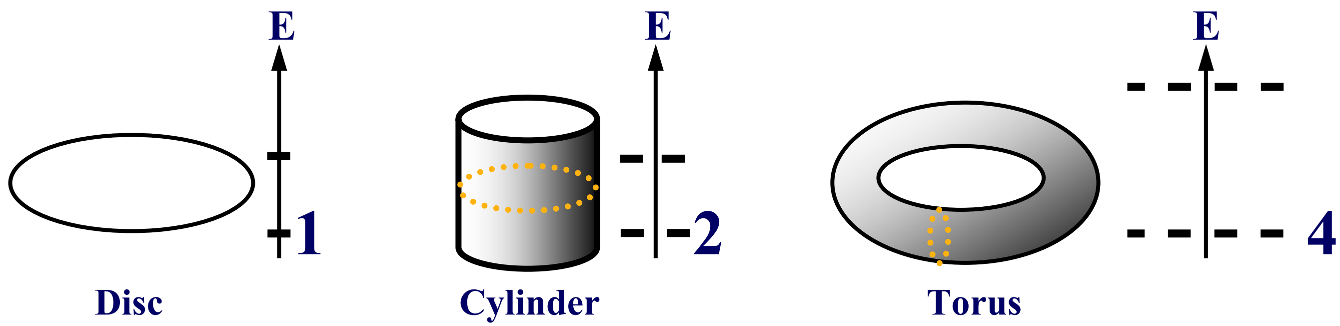
- **The ground state of "Rokhsar-Kivelson" Type RVB spin liquid is a Z2 topological order : all excitations are gapped; four-fold degeneracy on a torus**

D.S. Rokhsar and S. Kivelson, *Phys. Rev. Lett*. 61, 2376 (1988)
N. Read and S. Sachdev, *Phys. Rev. Lett*. 66, 1773 (1991);
X. G. Wen, *Phys. Rev*. B 44, 2664 (1991).

Topological degeneracy of RK model



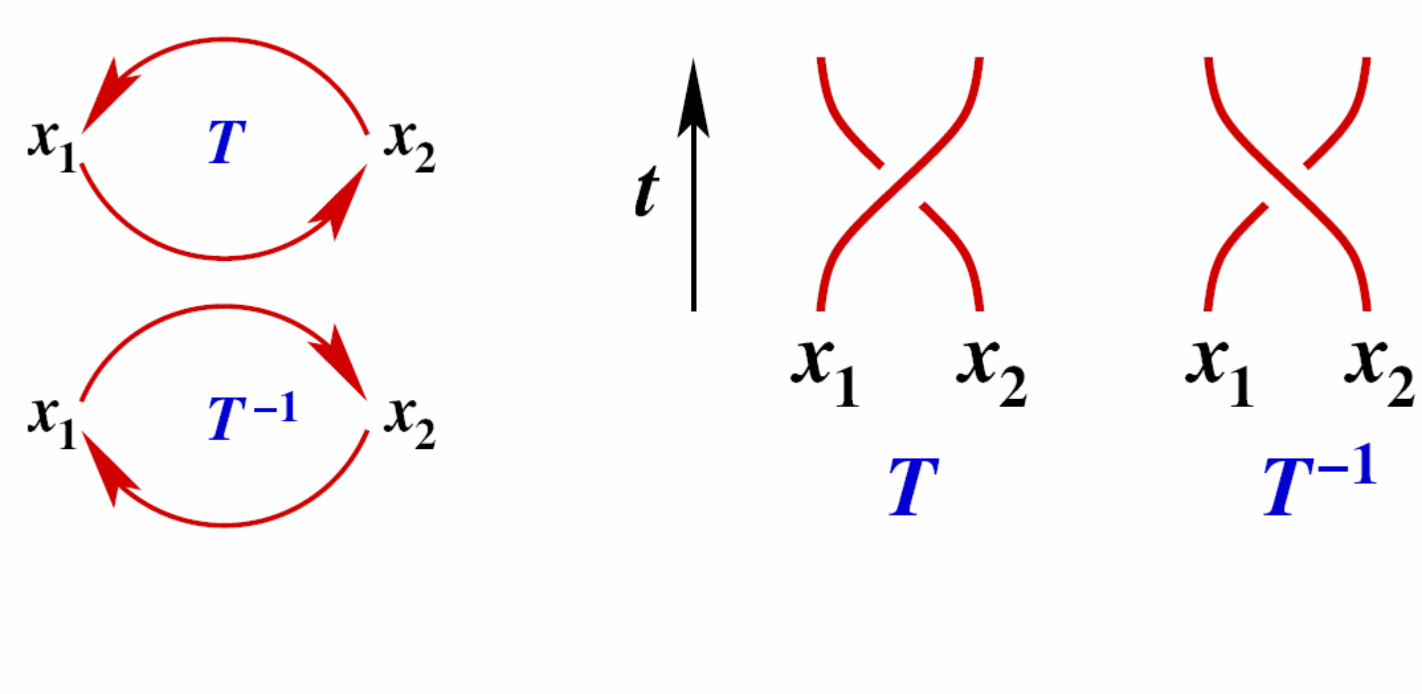




Ground-states are *locally* indistinguishable.

Degeneracy *robust* to local *perturbations*.

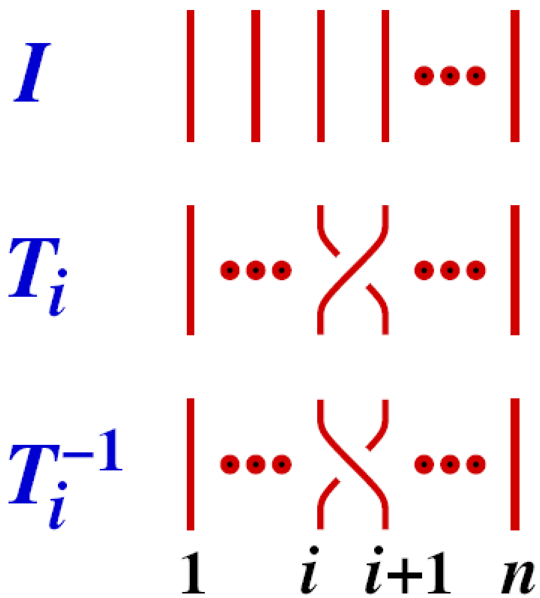
5. Exchange statistics in (2+1)D

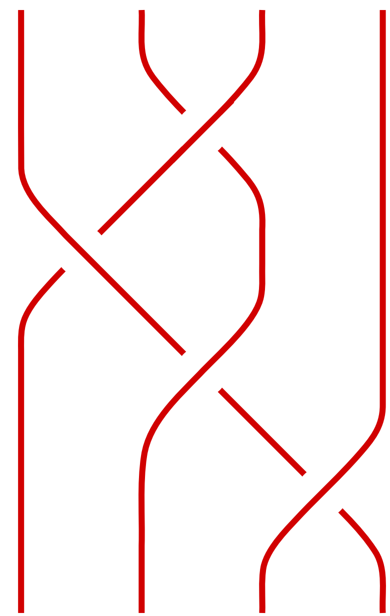


- In (3+1)D, $T^{-1} = T$, while in (2+1)D $T^{-1} \neq T$!
- If $T^{-1} = T$ then $T^2 = 1$, and the only types of particles are bosons and fermions.

The Braid Group

In (2+1)D, we should consider the braid group:

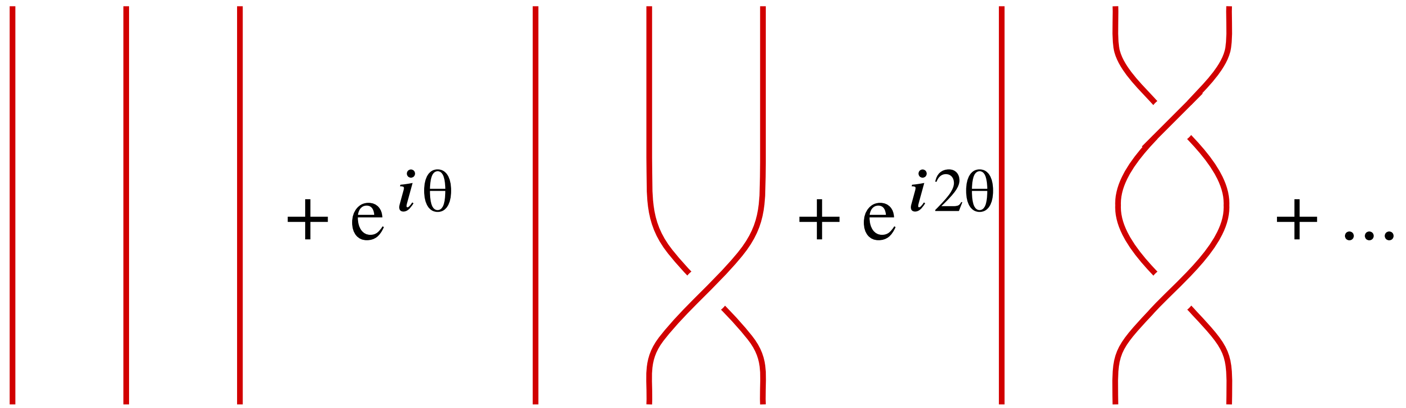




$T_3 T_2 T_1^{-1} T_2$

Abelian Anyons

- **Different elements of the braid group correspond to disconnected subspaces of trajectories in space-time.**
- **Possible choice: weight them by different overall phase factors (Leinaas and Myrrheim, Wilczek).**



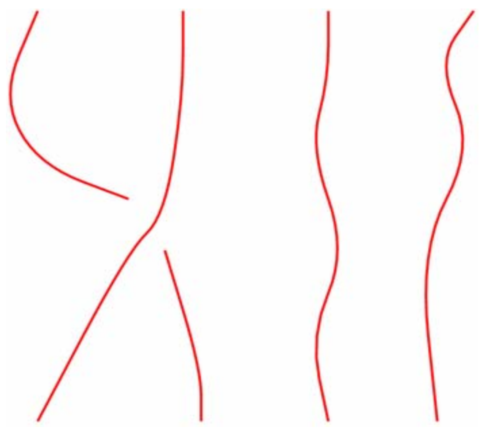
- **These phase factors realise an Abelian representation of the braid group.**

E.g $q = p/m$ for a Fractional Quantum Hall state at a filling factor $\nu = 1/m$.

- **The ground state degeneracy on a torus: m-fold degenerate ground states for FQHE (Haldane, Rezayi ’88, Wen ’90).**

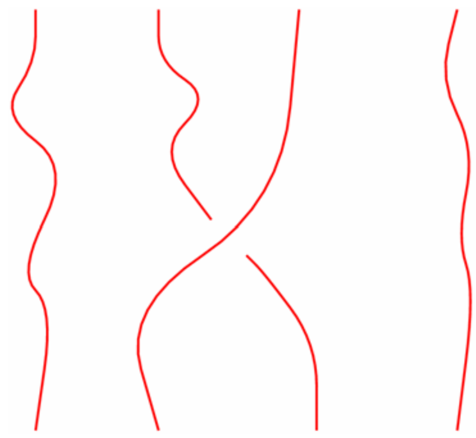
Non-Abelian Anyons

Exchanging particles 1 and 2:



$$\psi_a \rightarrow M^{12}_{ab}\psi_b$$

Exchanging particles 2 and 3:



$$\psi_a \rightarrow M^{23}_{ab}\psi_b$$

- **Matrices M^{12} and M^{23} need not commute, hence Non-Abelian Statistics.**
- **Matrices M form a higher-dimensional representation of the braid-group.**
- **For fixed particle positions, we have a non-trivial multi-dimensional Hilbert space where we can store information**

$\nu = 5/2$ **is believed to be MR = U(1)** $\times$ **Ising**

U(1) is a familiar Abelian factor due to electric charge

Ising particle types : I, σ, ψ

Fusion rules : $I \times I = I, \quad I \times \sigma = \sigma, \quad I \times \psi = \psi,$

$$\sigma \times \sigma = I + \psi, \quad \sigma \times \psi = \sigma, \quad \psi \times \psi = I$$

quasiholes carry anyonic charge : $(e/4, \sigma)$

electrons carry anyonic charge : $(-e, \psi)$

n **quasiholes carry anyonic charge :** $(ne/4, \sigma)$ **for** n **odd**

$(ne/4, I$ **or** $\psi)$ **for** n **even**

$\nu = 12/5$ **is believed to be** $\mathbf{RR_3 = U(1) \times Pf_3 = U(1) \times Fib}$

(Note : Fibonacci anyons can simulate a universal quantum computer.)

Fib particle types : I, ε

Fusion rules : $I \times I = I, \quad I \times \varepsilon = \varepsilon, \quad \varepsilon \times \varepsilon = I + \varepsilon$

quasiholes carry anyonic charge : $(e/5, \varepsilon)$

electrons carry anyonic charge : $(-e, I)$

n **quasiholes carry anyonic charge :** $(ne/5, I \text{ or } \varepsilon)$