Geometric versus Dynamical Gates in Quantum Computing

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Geometric phases (1)

- Adiabatic geometric phase (Berry):

When $H[R(0)] \to H[R(T)] = H[R(0)]$ adiabatically,

Eigenstate $|\xi_i\rangle \to e^{i(\alpha_i^d + \alpha_i^B)}|\xi_i\rangle$.

$\alpha_i^d = -\int E_i[R(t)]dt$ is the dynamical phase,

$\alpha_i^B = i\int dt\langle\xi_i|\partial_t\xi_i\rangle = i\int dR\langle\xi_i|\nabla_R\xi_i\rangle$ is the Berry phase.

Geometric phases (2)

- Aharonov-Anandan phase:

Cyclic in the projected space of rays:

$|\psi(0)\rangle\langle\psi(0)| \to |\psi(T)\rangle\langle\psi(T)| = |\psi(0)\rangle\langle\psi(0)|$

$$|\psi(0)\rangle \to e^{i\phi}|\psi(0)\rangle.$$

$\alpha^d = -\int \langle\psi(t)|H|\psi(t)\rangle dt$ is the dynamical phase,

$\alpha^g = \phi - \alpha^d$ is the AA or nonadiabatic geometric phase.

Geometric quantum computation

- In a certain basis, each basis state is transformed by a geometric phase.
- Thus the unitary transformation for an arbitrary state is realized in terms of geometric phases. I.e. for any state,

$$|\psi\rangle \to U|\psi\rangle,$$
$$U \text{ is geometric.}$$

Robustness

- Geometric phase is a global geometric property.
- Insensitive to decoherence.
- Insensitive to classical noise.

- Easier than topological quantum comp.

Previous discussions:

- no coupling between control qubit and external field.
- Ising type interaction for two-qubit operations.

- It seems that no universal set of quantum gates has been explicitly constructed for geometric QC.

[Only for standard universal sets, it is known how to actually implement quantum algorithms and quantum error orrections].

Standard universal set: {Hadamard, π /8, Controlled-NOT}.

We will:

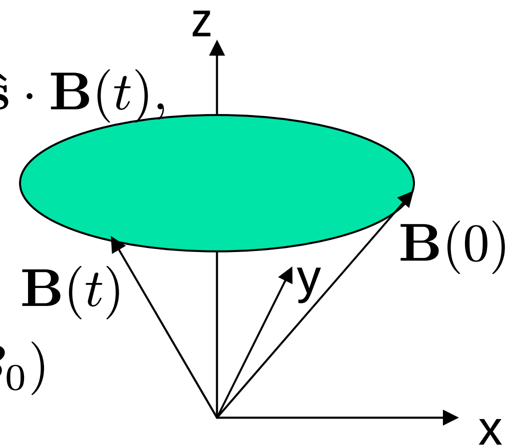
- Examine its implementation based on Zeeman coupling with a rotating field and isotropic Heisenberg interaction.
- Construct pi/8 and Hadamard gates using geometric phases.
- Show that it is impossible to construct a two-qubit gate based on Berry phases, or based on Aharonov-Anandan phases when the gyromagnetic ratios of the two qubits are equal.

A spin in an adiabatically rotating magnetic field

$$h = -\kappa[s_z B_0 + s_x B_1 \cos\phi(t) + s_y B_1 \sin\phi(t)] = -\kappa \hat{\mathbf{s}} \cdot \mathbf{B}(t),$$

$$\mathbf{B}(t) = [B_1 \cos\phi(t), B_1 \sin\phi(t), B_0]$$

$$B = \sqrt{B_0^2 + B_1^2}$$



Qubit basis: spin eigenstates along $\mathbf{B}(0) = (B_1, 0, B_0)$

$$\begin{aligned} |1_{\mathbf{B}}\rangle = |\uparrow_{\mathbf{B}}\rangle &= \cos\theta/2|\uparrow_z\rangle + \sin\theta/2 e^{i\phi}|\downarrow_z\rangle \\ |-1_{\mathbf{B}}\rangle = |\downarrow_{\mathbf{B}}\rangle &= \sin\theta/2|\uparrow_z\rangle - \cos\theta/2 e^{i\phi}|\downarrow_z\rangle, \end{aligned}$$

Subscript: quantization direction of the spin state. $\theta = \arctan\frac{B_0}{B_1}$

In a cycle, dynamical phase$= -\sigma_{\mathbf{B}(0)} B\tau/2$, $\quad \sigma_{\mathbf{B}(0)} = 1, -1$

Berry phase$= -\sigma_{\mathbf{B}(0)}\Omega(\theta)/2$, $\Omega(\theta) = 2\pi(1 - \cos\theta)$

Canceling the dynamical phase (spin echo)

Two consecutive cycles with opposite directions of $\mathbf{B}(0)$.

The dynamical phase is canceled, while the Berry phase Doubles.

For an arbitrary state,
the purely geometric single-qubit unitary transformation is

$$U_{1,ad} = diag(e^{i2\pi B_0/\sqrt{B_0^2+B_1^2}}, e^{-i2\pi B_0/\sqrt{B_0^2+B_1^2}}),$$

written in the basis $\{|\uparrow_{\mathbf{B}(0)}\rangle, |\downarrow_{\mathbf{B}(0)}\rangle\}$.

The phase is $-\sigma_{\mathbf{B}(0)}\Omega(\theta)$ up to $2\pi n$.

π /8 gate using Berry phases

$$T = diag(e^{-i\pi/8}, e^{i\pi/8})$$

$$U_{1,ad} = T \text{ when } B_0/|B_1| = -1/\sqrt{255}$$

Hadamard gate using Berry phases

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Qubit direction is still $(B_1, 0, B_0)$.

$\mathbf{B}(0)$ was rotated by an angle χ with an axix perpendicular to z. Using two loops with opposite signs of $\mathbf{B}(0)$, one obtains $U'_{1,ad} = diag(e^{i\alpha}, e^{-i\alpha})$, in the basis along the new direction of $\mathbf{B}(0)$.

In the qubit basis, $U'_{1,ad}$ is

$$\begin{pmatrix} e^{i\alpha}\cos^2\frac{\chi}{2} + e^{-i\alpha}\sin^2\frac{\chi}{2} & i\sin\alpha\sin\chi \\ i\sin\alpha\sin\chi & e^{i\alpha}\cos^2\frac{\chi}{2} + e^{-i\alpha}\sin^2\frac{\chi}{2} \end{pmatrix}.$$

$$U'_{1,ad} = iH \text{ when } \chi = \pi/4,\ \alpha = \pi/2.$$

Aharonov-Anandan phase gates

The Hamiltonian is the same above, but:
It is specified that $\phi(t) = \omega t$.
No adiabatic condition.

$$|\psi(t)\rangle = e^{-is_z \omega t} e^{-i\tilde{h}t} |\psi(0)\rangle,$$

$\tilde{h} = h(0) - \omega s_z = -\kappa \tilde{\mathbf{B}} \cdot \mathbf{s}$
is **time-independent**,
$\tilde{\mathbf{B}} = (B_1, 0, B_0 + \omega/\kappa)$. $\tilde{B} = \sqrt{(B_0 + \omega/\kappa)^2 + B_1^2}$.

Qubit basis state: $|\sigma_{\tilde{\mathbf{B}}}\rangle$

Phases

After period $\tau = 2\pi/\omega$,

$|\sigma_{\tilde{\mathbf{B}}}\rangle \to e^{i\alpha}|\sigma_{\tilde{\mathbf{B}}}\rangle$

Total phase $\alpha = \pi + \kappa\sigma_{\tilde{\mathbf{B}}}\tilde{B}\tau/2$,

$\alpha^{d} = \sigma_{\kappa\tilde{\mathbf{B}}}(\tilde{B}\tau/2 - \pi\cos\tilde{\theta})$,

$\alpha^{aa} = -\sigma_{\tilde{\mathbf{B}}}\Omega(\tilde{\theta})/2$, up to $2\pi n$,

$\Omega(\tilde{\theta}) = 2\pi(1 - \cos\tilde{\theta})$ is the solid angle subtended by $\mathbf{s}$.

Single-qubit geometric gates

The dynamical phase vanishes when $B_0 = [-\omega/\kappa \pm \sqrt{(\omega/\kappa)^2 - 4B_1^2}]/2.$

$$U_{1,aa} = diag(e^{i\pi(B_0+\omega/\kappa)/\sqrt{(B_0+\omega/\kappa)^2+B_1^2}}, e^{-i\pi(B_0+\omega/\kappa)/\sqrt{(B_0+\omega/\kappa)^2+B_1^2}}).$$

$$U_{1,aa} = T \text{ when } (B_0 + \omega/\kappa)/|B_1| = -1/\sqrt{63}.$$

Hadamard gate

Qubit basis still defined by $(B_1, 0, B_0 + \omega/\kappa)$.
$\tilde{\mathbf{B}}$ is rotated from $(B_1, 0, B_0 + \omega/\kappa)$
by an angle χ with z aixis (achieved by rotating $\mathbf{B}$).

Under condition $B_0 = [-\omega/\kappa \pm \sqrt{(\omega/\kappa)^2 - 4B_1^2}]/2$.
one can obtain purely geometric gate using AA phases,
$U_{1,aa}$, written in the basis in the direction of $\tilde{\mathbf{B}}(0)$.
iH (in the qubit basis) can be obtained if $\chi = \pi/4$, $(B_0 + \omega/\kappa)/|B_1| = 1/\sqrt{3}$.

Additional aspects of robustness

1. For the two consecutive cycles,
only the period needs to be the same,
while the time-dependence of $\phi(t)$
in each cycle can be completely arbitrary.
The underlying reason:
the energy for each instantaneous eigenstate
is time-independent,
thus the dynamical phase is simply a product of
the energy and the period.

2. The two-cycle method of canceling the field-caused dynamical phases are the same for all gates.

Additional aspects of robustness

2. The two-cycle method of canceling the field-caused dynamical phases are the same for all gates.

Two interacting spins

$$\mathcal{H} = h_a + h_b + J\mathbf{s}_a \cdot \mathbf{s}_b$$

$$h_j = -\kappa_j \hat{\mathbf{s}}_j \cdot \mathbf{B}(t)$$

The instantaneous eigenstates:

$$|\xi_1(t)\rangle = |\uparrow^{\alpha}_{\mathbf{B}(t)}\uparrow^{\beta}_{\mathbf{B}(t)}\rangle,$$

$$|\xi_{2/3}(t)\rangle = \frac{1}{\mathcal{N}_{\pm}}\{[-(\kappa_\alpha-\kappa_\beta)\frac{B}{J}\pm\sqrt{(\kappa_\alpha-\kappa_\beta)^2\frac{B^2}{J^2}+1}]|\uparrow^{\alpha}_{\mathbf{B}(t)}\downarrow^{\beta}_{\mathbf{B}(t)}\rangle+|\downarrow^{\alpha}_{\mathbf{B}(t)}\uparrow^{\beta}_{\mathbf{B}(t)}\rangle\},$$

$$|\xi_4(t)\rangle = |\downarrow^{\alpha}_{\mathbf{B}(t)}\downarrow^{\beta}_{\mathbf{B}(t)}\rangle$$

Eigenvalues: $E_1 = -\frac{1}{2}(\kappa_\alpha+\kappa_\beta)B+\frac{1}{4}J$, $E_{2/3} = -\frac{J}{4}\pm\frac{1}{2}\sqrt{(\kappa_\alpha-\kappa_\beta)^2B^2+J^2}$, $E_4 = \frac{1}{2}(\kappa_\alpha+\kappa_\beta)B+\frac{J}{4}$.

Phases

When $\phi \to \phi + 2\pi$, $|\xi_i\rangle \to e^{i(\gamma_i^d + \gamma_i^B)}|\xi_i\rangle$.
$\alpha_i^d = -E_i \tau$, τ is the cycling time.

Berry phases:
$\gamma_1^b = \frac{2\pi B_0}{\sqrt{B_0^2 + B_1^2}}$,
$\gamma_2^b = \gamma_3^b = 0$,
$\gamma_4^b = -\frac{2\pi B_0}{\sqrt{B_0^2 + B_1^2}}$.
They are exactly equal to $-(s^\alpha_{\mathbf{B}(t)} + s^\beta_{\mathbf{B}(t)})$ multiplied by the solid angle $\Omega(\theta)$, up to $2\pi n$.

Unitary transformation

$$U_{2,ad} = diag(e^{-iE_1\tau + i\Omega(\theta)}, e^{-iE_2\tau}, e^{-iE_3\tau}, e^{-iE_4\tau - i\Omega(\theta)})$$

in the basis $\{\xi_1, \xi_2, \xi_3, \xi_4\}$.

Geometric part of the Unitary transformation

$diag(e^{i\Omega(\theta)}, 1, 1, e^{-i\Omega(\theta)})$,

which retains the same form in the qubit basis

$\{|\uparrow_{\mathbf{B}(0)}\uparrow_{\mathbf{B}(0)}\rangle, |\uparrow_{\mathbf{B}(0)}\downarrow_{\mathbf{B}(0)}\rangle, |\downarrow_{\mathbf{B}(0)}\uparrow_{\mathbf{B}(0)}\rangle, |\downarrow_{\mathbf{B}(0)}\downarrow_{\mathbf{B}(0)}\rangle\}$,

can be factorized as $(e^{i\Omega(\theta)/2}, e^{-i\Omega(\theta)/2}) \otimes (e^{i\Omega(\theta)/2}, e^{-i\Omega(\theta)/2})$

a product of single-qubit transformations!

Therefore, two-qubit gates cannot be constructed based purely on Berry phases.

Similar is the situation for AA phases based on the same interaction, in case $\kappa_a = \kappa_b$.

Summary

- Using Zeeman coupling with a rotating field, we have constructed standard single qubit quantum gates based on Berry and *AA* phases respectively.
- Adding Heisenberg interaction, one cannot construct two-qubit gates purely based on Berry phases, or purely based on *AA* phases in case the gyromagnetic ratios of the two qubits are equal.
- One may combine single qubit geometric gates and two-qubit dynamical gates.

Thank you for your attention!