Anomalous Monopole in an Interacting Boson System

WU Biao（吴飙）

Institute of Physics, Chinese Academy of Sciences

September 15th, 2008 at Zhengzhou

Acknowledgements

Collaborators: ZHANG Qi(张起)
LIU Jie(刘杰)

¥: **BaiRen program of CAS**
NSF of China
the 973 projects of China

Thank CAO Zexian for translating von Neumann and Wigner's paper from German to English

Reference: arXiv:0802.2600

Brief History of Monopole

P.A.M. Dirac, *Proc. of R. Soc. of London* **133** (1931)60

$$\frac{2ge}{\hbar c} = n \quad \Rightarrow \quad \text{If } n = 1,\ g = \frac{\hbar c}{2e^2} e = \frac{137}{2} e$$

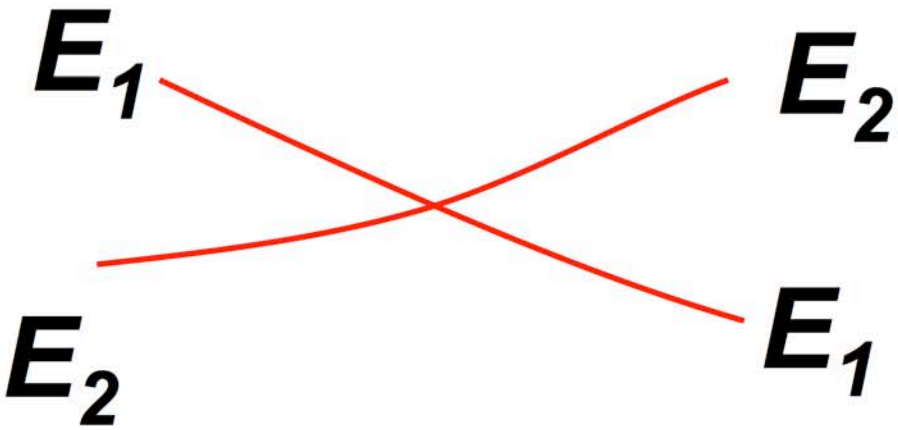
One experimental report (PRL 1982)

exists in many GUTs with enormously large mass (>10^{16}GeV)

Monopole as degeneracy point

Hamiltonian $H(R_1, R_2, \ldots, R_k, \ldots)$,

eigenstates $E_n(R_1, R_2, \ldots, R_k, \ldots)$



- **degeneracy due to symmetries**
- **accidental degeneracy (monopole)**

Berry phase and monopole

Proc. R. Soc. Lond. A **392**, 45–57 (1984)
Printed in Great Britain

Quantal phase factors accompanying adiabatic changes

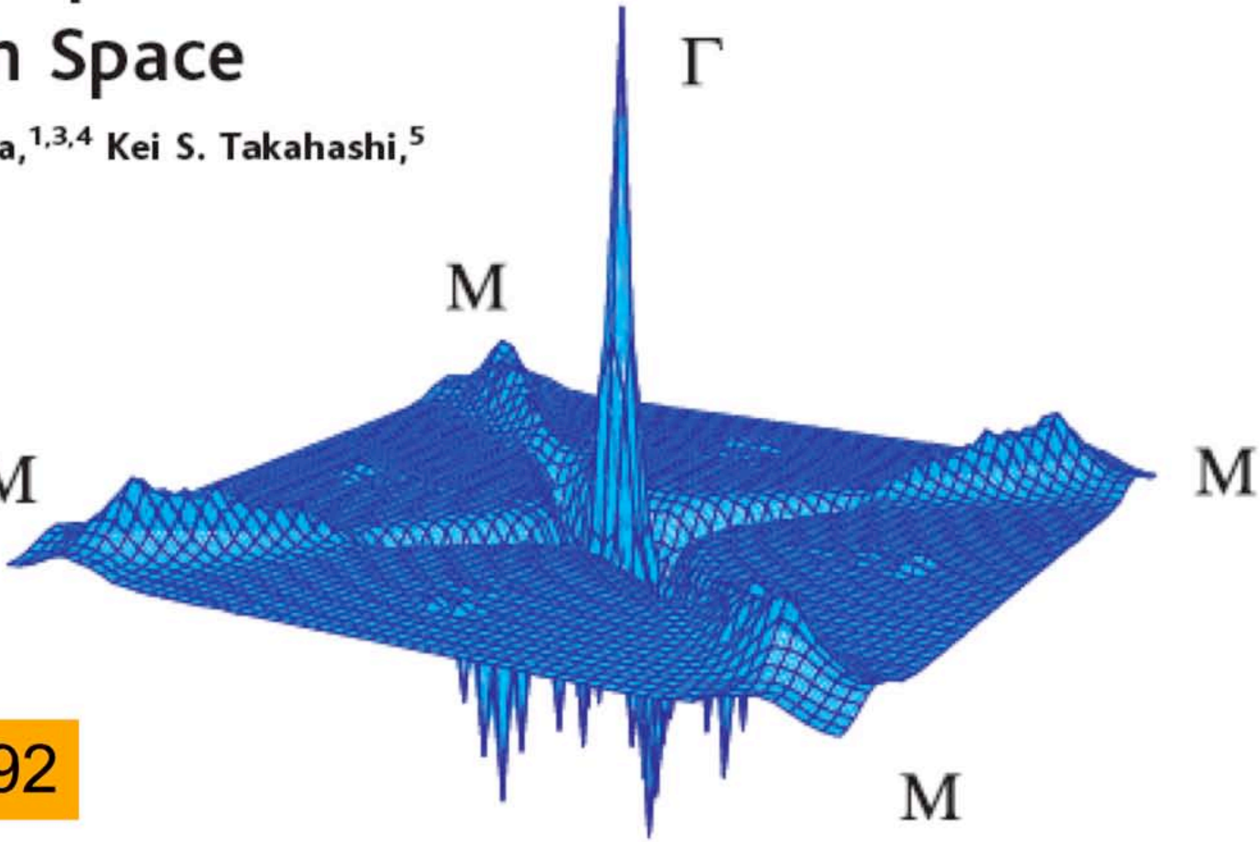
BY M. V. BERRY, F.R.S.

- the phase change $\gamma_+(C)$ is the flux through C of the magnetic field of a monopole with strength $-\frac{1}{2}$ located at the degeneracy.
- $\gamma_n(C)$ is the flux through C of the 'magnetic field' of a monopole $-n$ located at the origin of magnetic field space.
- $\boldsymbol{V}(\boldsymbol{R})$ would be the sum of the 'magnetic fields' of 'monopoles' situated at the degeneracies

Monopole in *k* space

The Anomalous Hall Effect and Magnetic Monopoles in Momentum Space

Zhong Fang,[1,2]* Naoto Nagaosa,[1,3,4] Kei S. Takahashi,[5]



Science **302** (2003)92

Monopoles in comparision

as a fundamental particle	as a degeneracy point
magnetic field	Berry curvature
charge: $n\left(\frac{137}{2}e\right)$	Chern number: $n(2\pi)$
mass	no mass
Other physical properties	none

Von Neumann-Wigner Theorem
for physicists

Consider a *nxn* Hermitian matrix. It has n^2 parameters and *n eigenvalues* $E_1, E_2, E_3 \ldots E_n$

How many parameters do you need to change to get a degeneracy ($E_j=E_k=\ldots$)?

Von Neumann-Wigner theorem:

***3* parameters for double degeneracy**

***8* parameters for triple degeneracy**

m^2-*1* parameters for *m*-fold degeneracy

von Neumann and Wigner, Phys. Z. **30**(1929) 467

Von Neumann-Wigner Theorem
for mathematicians

Consider a $n x n$ Hermitian matrix. It has n^2 parameters and n *eigenvalues* $E_1, E_2, E_3 \dots E_n$

What is the codimension of space of degeneracy points?

Von Neumann-Wigner theorem:

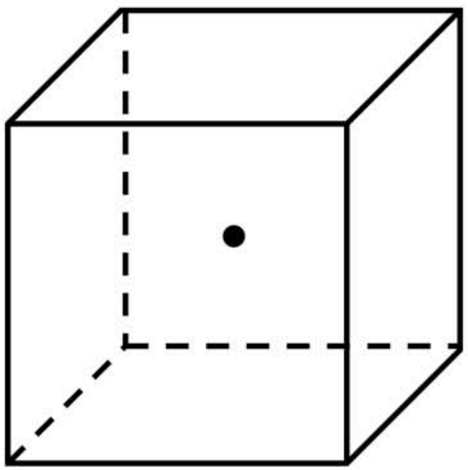
Codimension for double degeneracy: 3
Codimension for triple degeneracy: 8
Codimension for *m*-fold degeneracy: m^2-1

von Neumann and Wigner, Phys. Z. **30**(1929) 467

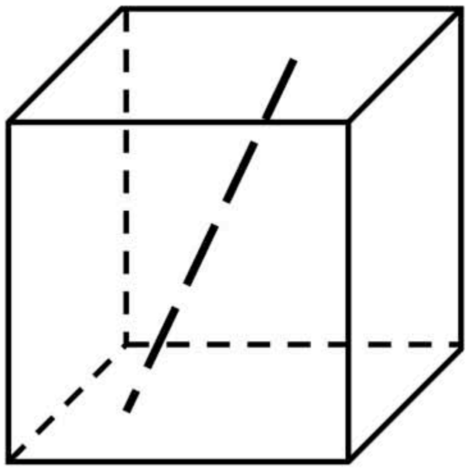
Codimension

If W is a linear subspace of a vector space V, then the codimension of W in V is the dimension of the quotient space V/W.

$$\text{codim}(W) = \dim(V) - \dim(W).$$



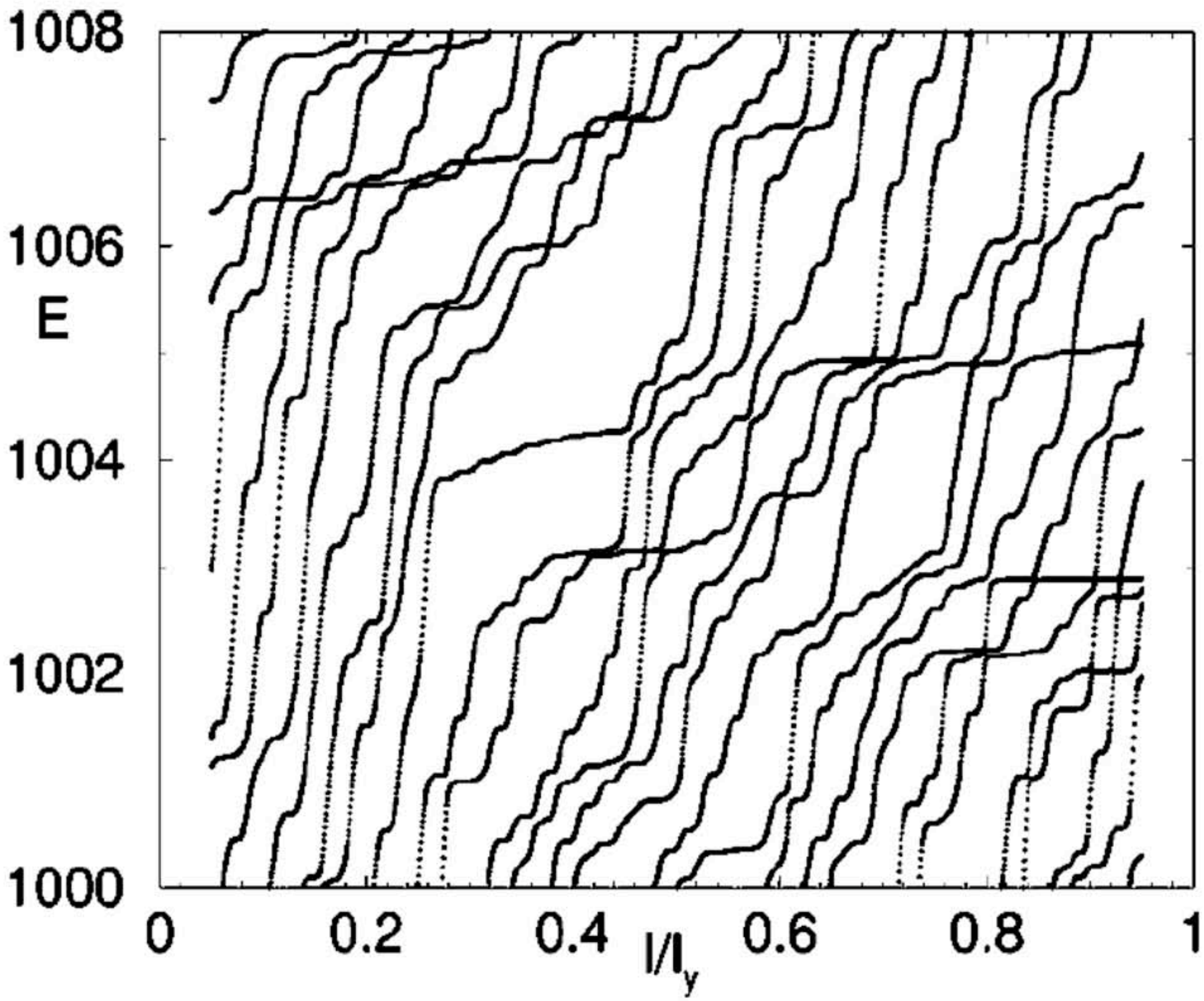
codim(*point*)=3



codim(*line*)=2

Avoided crossing of energy levels

implied in von Neumann-Wigner theorem



2x2 Hermitian matrix

$$H = \begin{pmatrix} R_z + E_0 & R_x + iR_y \\ R_x - iR_y & -R_z + E_0 \end{pmatrix}$$

Eigenvalues:

$$E_{\pm} = E_0 \pm \sqrt{R_x^{\ 2} + R_y^{\ 2} + R_z^{\ 2}}$$

They become degenerate only at $R_x=R_y=R_z=0$; therefore, degeneracy points make up a line in a 4-dimensional space or the codimension of degeneracy points is 3.

For physicists, E_0 is trivial and can be taken as zero. As a result, the line of degeneracy becomes an isolated point in a 3D space.

Monopoles in 3D must be points

Consider a quantum system, whose Hamiltonian depends on three parameters *H(X,Y,Z)*. At a double degeneracy point, *H(X,Y,Z)* can be approximated by a 2x2 Hermitian matrix

$$H = \begin{pmatrix} R_z(X,Y,Z) & R_x(X,Y,Z) + iR_y(X,Y,Z) \\ R_x(X,Y,Z) + iR_y(X,Y,Z) & -R_z(X,Y,Z) \end{pmatrix}$$

$$R_x, R_y, R_z \xleftrightarrow{\text{1-1 map}} X, Y, Z$$

The degeneracy of *H(X,Y,Z)* occurs only at isolated points in parameter spaces.

Interacting two-mode boson system

$$
\begin{aligned}
\hat{H}_N = & \frac{X}{2}(\hat{a}^\dagger\hat{b}+\hat{a}\hat{b}^\dagger)+\frac{iY}{2}(\hat{a}\hat{b}^\dagger-\hat{a}^\dagger\hat{b})+ \\
& \frac{Z}{2}(\hat{a}^\dagger\hat{a}-\hat{b}^\dagger\hat{b})-\frac{\lambda}{4V}(\hat{a}^\dagger\hat{a}-\hat{b}^\dagger\hat{b})^2,
\end{aligned}
$$

Total number of bosons is *N*

Interacting two-mode boson system

$$\hat{H}_N = \frac{X}{2}(\hat{a}^\dagger\hat{b} + \hat{a}\hat{b}^\dagger) + \frac{iY}{2}(\hat{a}\hat{b}^\dagger - \hat{a}^\dagger\hat{b}) + \frac{Z}{2}(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}) - \frac{\lambda}{4V}(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})^2,$$

Total number of bosons is *N*

$$\hat{J}_x = (\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a})/2 \quad \hat{J}_y = i(\hat{a}^\dagger\hat{b} - \hat{b}^\dagger\hat{a})/2 \quad \hat{J}_z = (\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})/2$$

Hamiltonian of an easy-axis magnetic particle

$$\hat{H}_N = X\hat{J}_x + Y\hat{J}_y + Z\hat{J}_z - \frac{\lambda}{V}\hat{J}_z^2$$

Mean-field theory

When *N* is large, we have

$$\hat{a} \rightarrow \sqrt{N}a + \delta\hat{a} \quad \hat{b} \rightarrow \sqrt{N}b + \delta\hat{b}$$

the Hamiltionian becomes

$$\begin{aligned} H_s &= \lim_{N\to\infty} \frac{\hat{H}_N}{N} = \frac{X}{2}(a^*b + ab^*) + \frac{iY}{2}(ab^* - a^*b) \\ &+ \frac{Z}{2}(|a|^2 - |b|^2) - \frac{c}{4}(|a|^2 - |b|^2)^2, \end{aligned}$$

$$|a|^2 + |b|^2 = 1, \quad c = N\lambda/V$$

Large *N* limit is semiclassical limit

At large *N* limit, we have $\langle \hat{a} \rangle \sim \sqrt{N},\ \langle \hat{b} \rangle \sim \sqrt{N}$

This means that $\left[\hat{a}, \hat{a}^{+}\right] = 1,\quad \left[\hat{b}, \hat{b}^{+}\right] = 1$

can be approximated as $\left[\hat{a}, \hat{a}^{+}\right] = 0,\quad \left[\hat{b}, \hat{b}^{+}\right] = 0$

Introduce $\hat{a}' = \hat{a} / \sqrt{N},\ \hat{b}' = \hat{b} / \sqrt{N}$

we have $\left[\hat{a}', \hat{a}'^{+}\right] = 1/N,\quad \left[\hat{b}', \hat{b}'^{+}\right] = 1/N$

Large *N* limit is semiclassical limit

At large *N* limit, we have $\langle \hat{a} \rangle \sim \sqrt{N}, \ \langle \hat{b} \rangle \sim \sqrt{N}$

This means that $\left[\hat{a}, \hat{a}^{+}\right] = 1, \quad \left[\hat{b}, \hat{b}^{+}\right] = 1$

can be approximated as $\left[\hat{a}, \hat{a}^{+}\right] = 0, \quad \left[\hat{b}, \hat{b}^{+}\right] = 0$

Reviews of Modern Physics, Vol. 54, No. 2, April 1982

Large *N* limits as classical mechanics

Laurence G. Yaffe*

California Institute of Technology, Pasadena, California 91125

Mean-field theory is a semiclassical theory

$$\hat{H}_N = \frac{X}{2}(\hat{a}^\dagger\hat{b} + \hat{a}\hat{b}^\dagger) + \frac{iY}{2}(\hat{a}\hat{b}^\dagger - \hat{a}^\dagger\hat{b}) + \frac{Z}{2}(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}) - \frac{\lambda}{4V}(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})^2,$$

semiclassical limit ⬇ **large *N* limit**

$$H_{\mathrm{s}} = \lim_{N\to\infty}\frac{\hat{H}_N}{N} = \frac{X}{2}(a^*b + ab^*) + \frac{iY}{2}(ab^* - a^*b) + \frac{Z}{2}(|a|^2 - |b|^2) - \frac{c}{4}(|a|^2 - |b|^2)^2,$$

Quantum Ground state

$$\hat{H}_N = \frac{X}{2}(\hat{a}^\dagger\hat{b} + \hat{a}\hat{b}^\dagger) + \frac{iY}{2}(\hat{a}\hat{b}^\dagger - \hat{a}^\dagger\hat{b}) + \frac{Z}{2}(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}) - \frac{\lambda}{4V}(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})^2,$$

At *X=Y=Z=0* $\hat{H}_N = -\frac{\lambda}{4V}(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})^2$

which has two ground states $|N,0\rangle$ and $|0,N\rangle$

The system's ground state is doubly degenerate at an isoalted point *X=Y=Z=0.*
Or the system has a monopole at *X=Y=Z=0*

Mean-field ground state

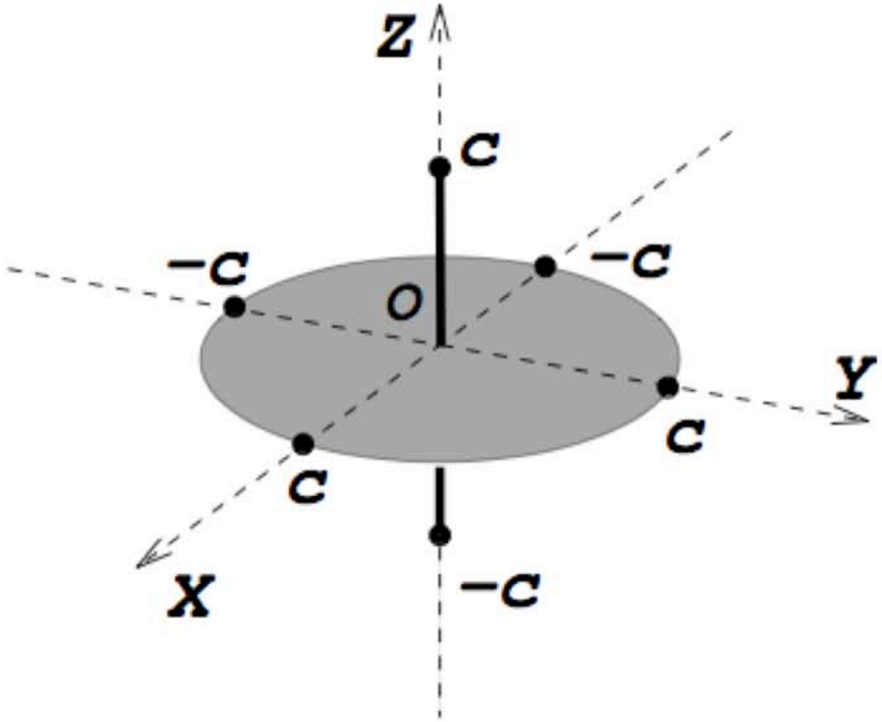
$$|\phi\rangle \equiv \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1-p}{2}} \\ -\sqrt{\frac{1+p}{2}}\frac{X+iY}{\sqrt{X^2+Y^2}} \end{pmatrix}$$

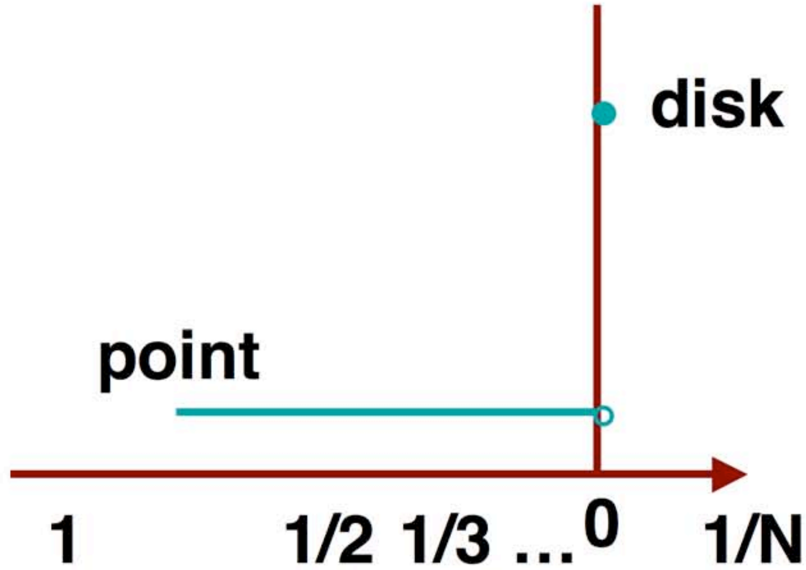
where p is the solution of the following equation,

$$p\sqrt{X^2+Y^2} = (Z+cp)\sqrt{1-p^2}.$$

- **When** $\sqrt{X^2+Y^2} \geq c$**, only one real root**
- **When** $X^2+Y^2 < c^2$ **and** $Z=0$**, three real roots with two of them corresponding to the ground state with identical energy**

Monopole of disk shape





Failure of von Neumann-Wigner theorem in the semiclassical limit

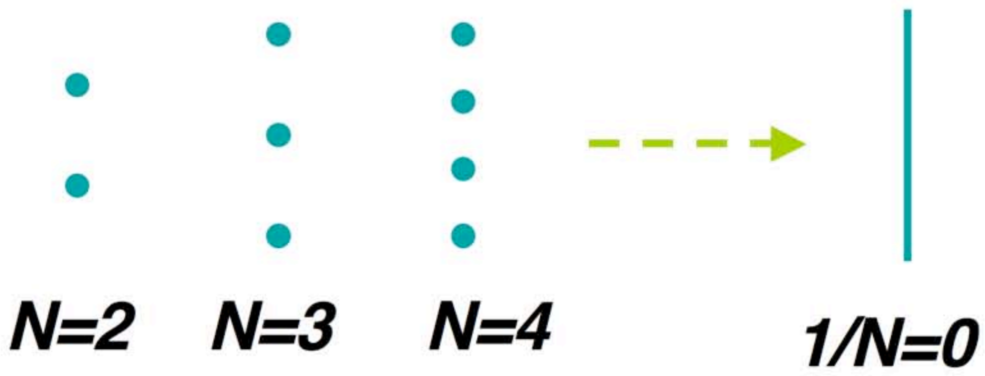
Highest quantum eigenstates

At *X=Y=0*,

$$\hat{H}_N = \frac{Z}{2}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}) - \frac{\lambda}{4V}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})^2$$

Its highest eigenstates are doubly degenerate at

$$\frac{Z}{c} = \frac{K}{N}, \qquad (K = -N+1, -N+3, \ldots, N-3, N-1)$$



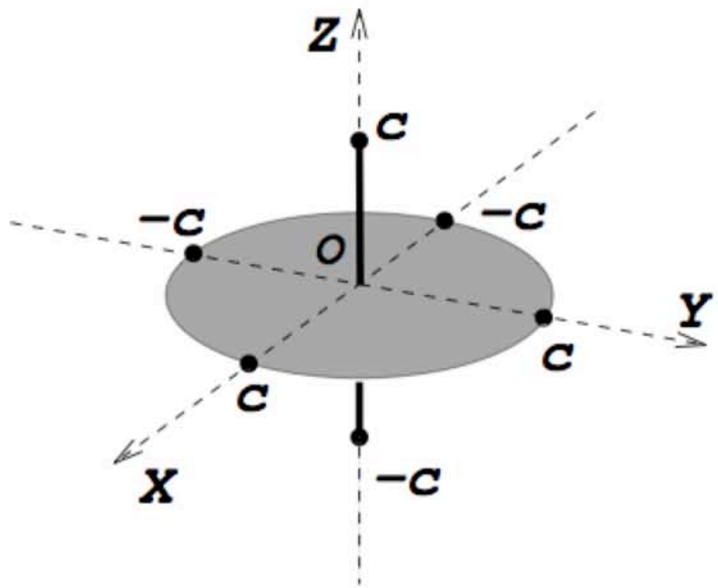
Mean-field highest eigenstate

At *X=Y=0*, the mean-field Hamiltonian is

$$H_s = \frac{Z}{2}(|a|^2 - |b|^2) - \frac{c}{4}(|a|^2 - |b|^2)^2$$

On the line *-c < Z < c*, the highest eigenstate

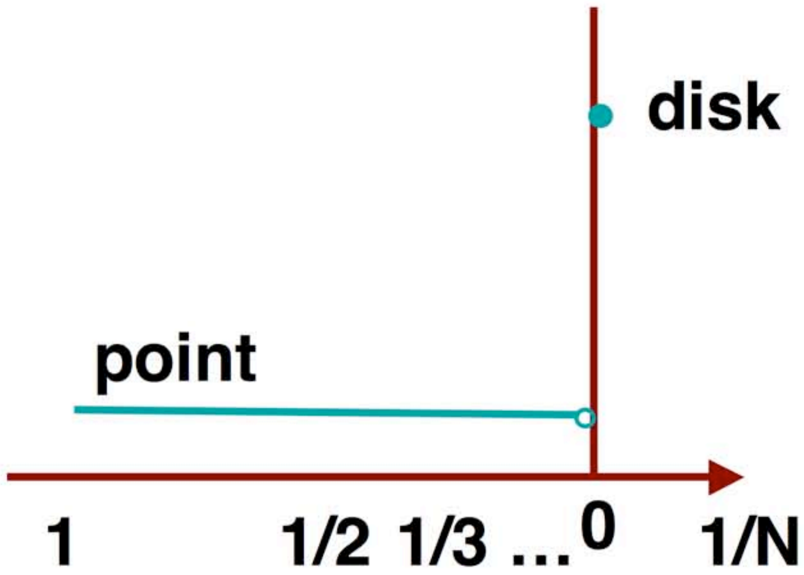
$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} e^{i\varphi}\sqrt{\dfrac{c+Z}{2c}} \\ \sqrt{\dfrac{c-Z}{2c}} \end{pmatrix}$$



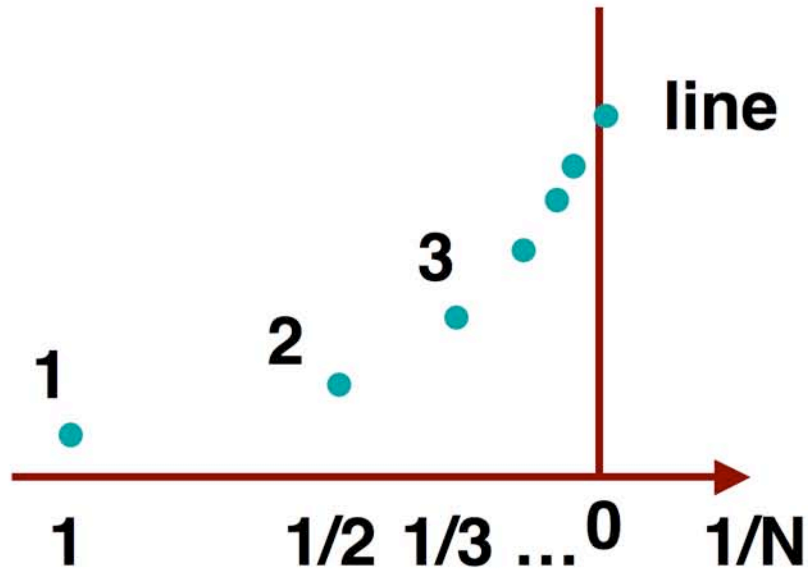
which is highly degenerate.

Disk Monopole vs. Line Monopole

Ground state

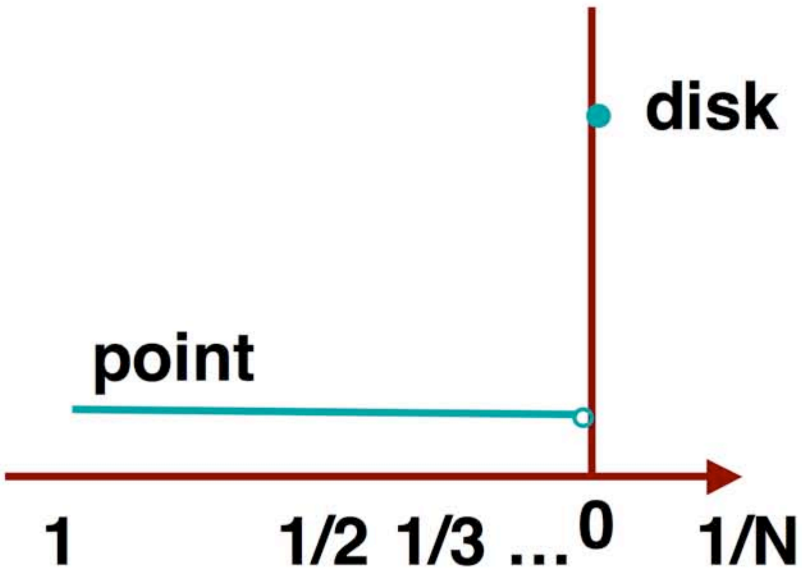


Highest eigenstate

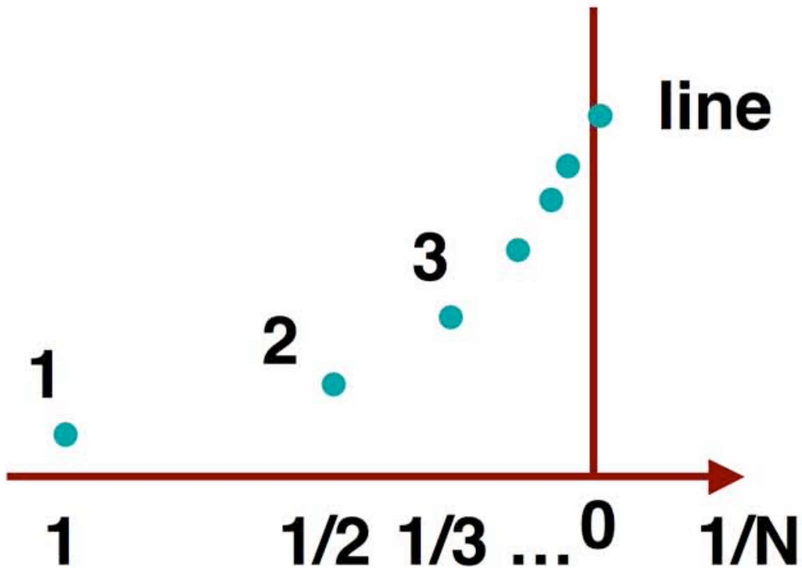


Disk Monopole vs. Line Monopole

Ground state



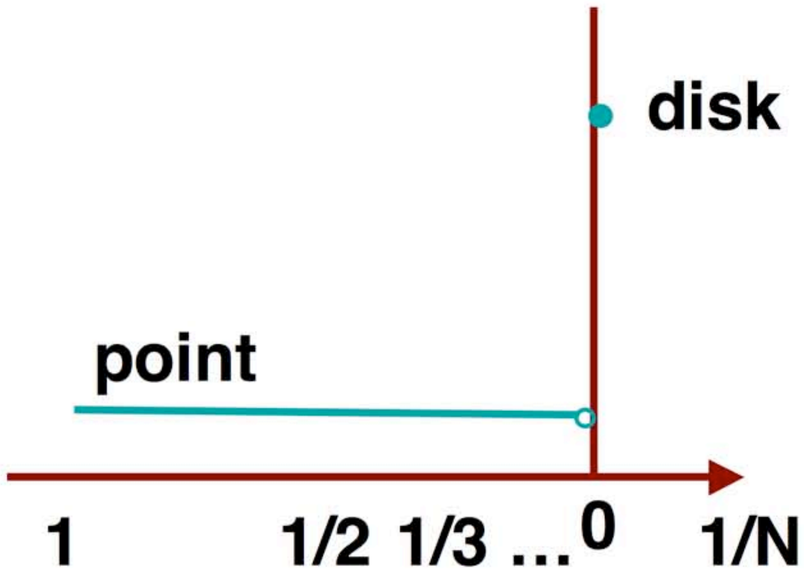
Highest eigenstate



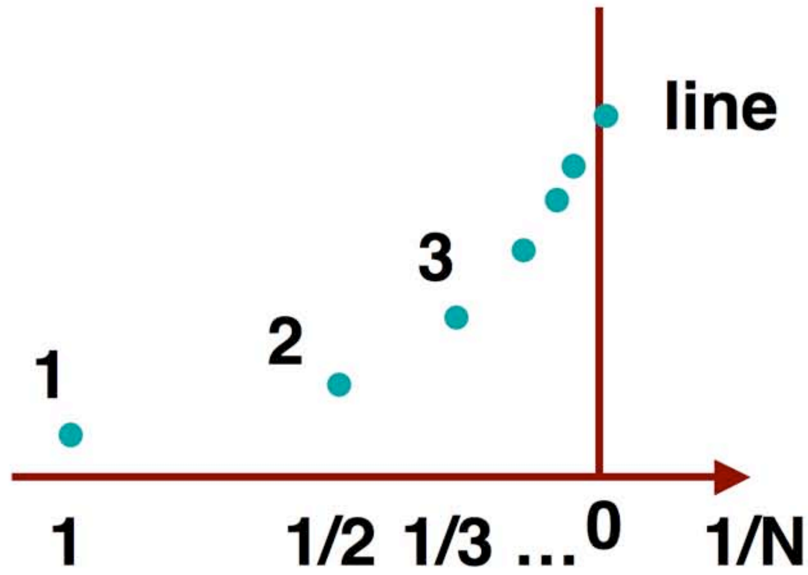
trivial

Disk Monopole vs. Line Monopole

Ground state



Highest eigenstate



?

trivial

Berry curvature

Hamiltonian $H(X,Y,Z)$

Eigenstate $\left|\psi_n(X,Y,Z)\right\rangle$

Vector potential

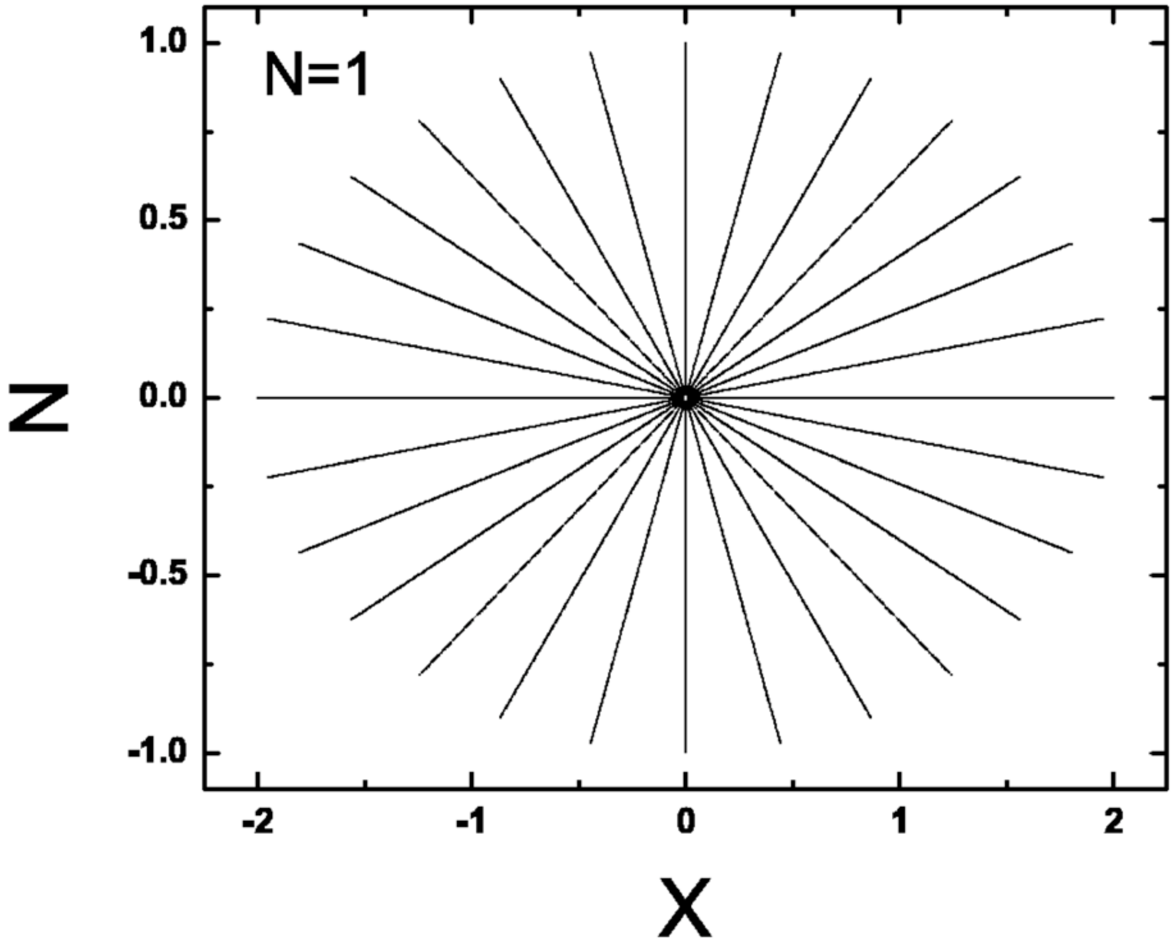
$$\vec{A}_n = \left\langle \psi_n(X,Y,Z) \middle| \nabla \middle| \psi_n(X,Y,Z) \right\rangle$$

Berry curvature: magnetic field in parameter space

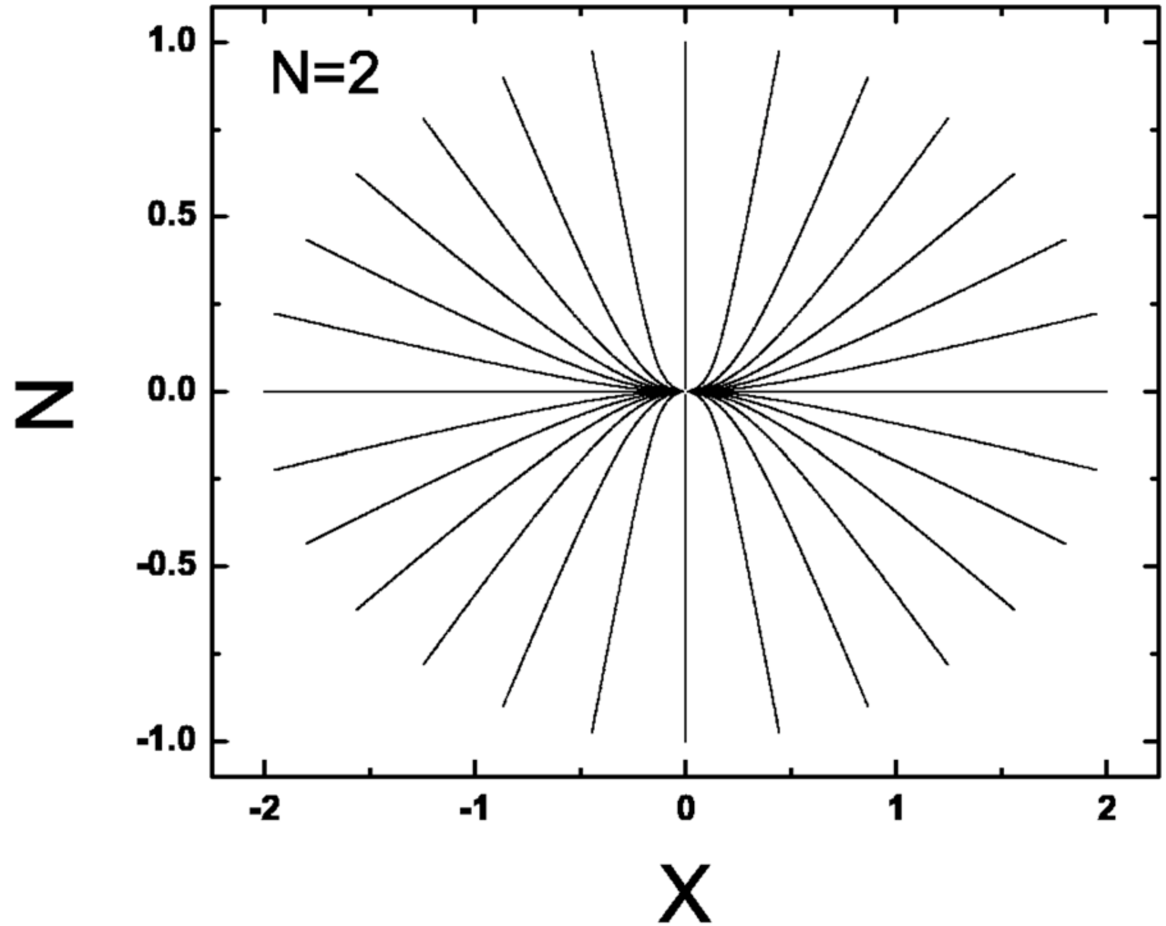
$$\vec{B}_n = \nabla \times \vec{A}_n = \nabla \times \left\langle \psi_n(X,Y,Z) \middle| \nabla \middle| \psi_n(X,Y,Z) \right\rangle$$

Field lines of Quantum Berry curvature

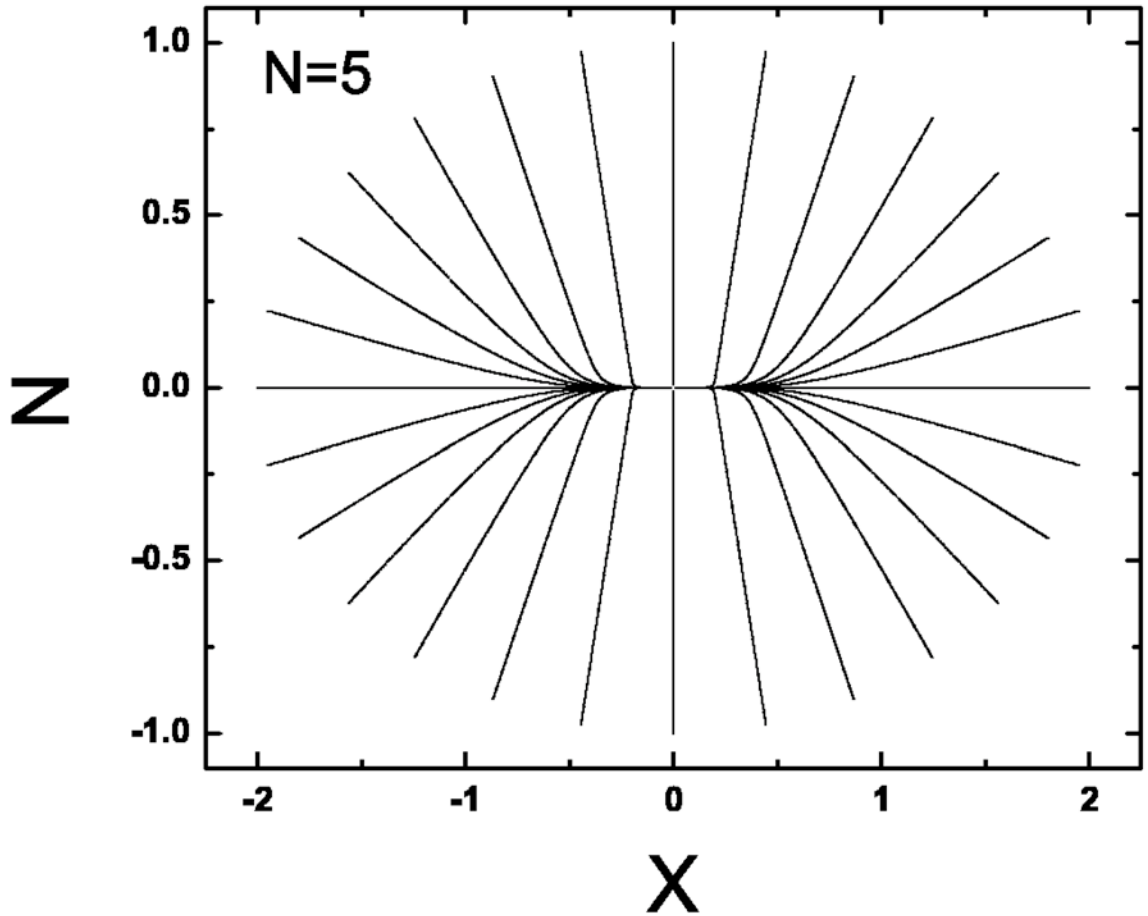
Field lines of Quantum Berry curvature



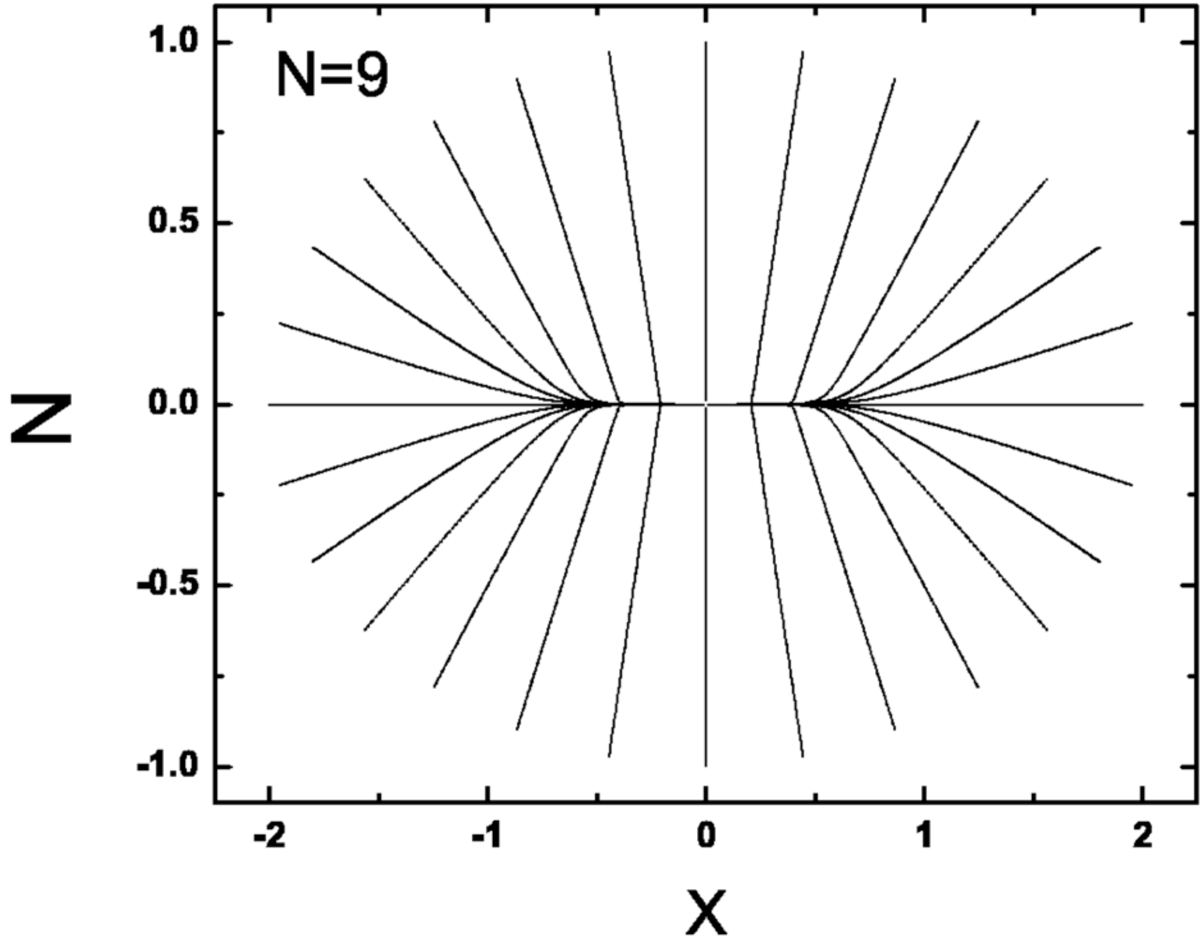
Field lines of Quantum Berry curvature



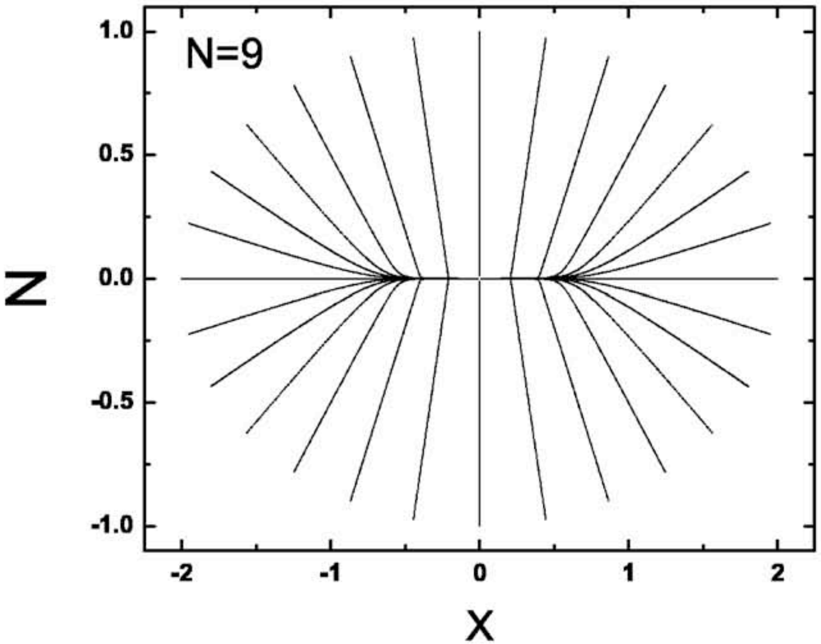
Field lines of Quantum Berry curvature



Field lines of Quantum Berry curvature



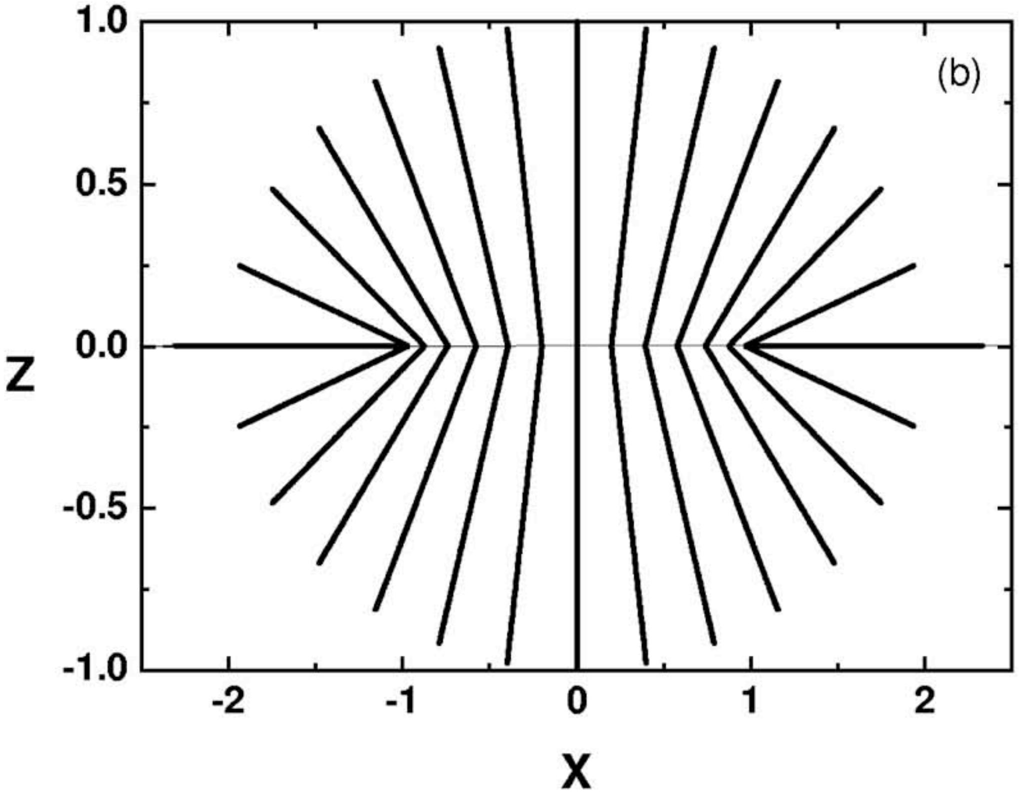
Quantum origin of the disk-shaped monopole

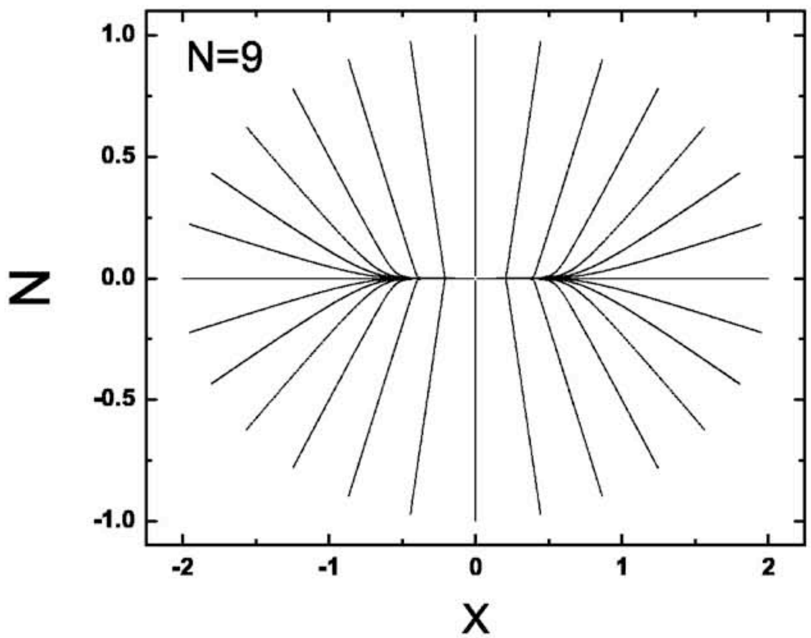


Disk-shaped monopole arises as the result of collapsing or bunching of field lines of Berry curvature

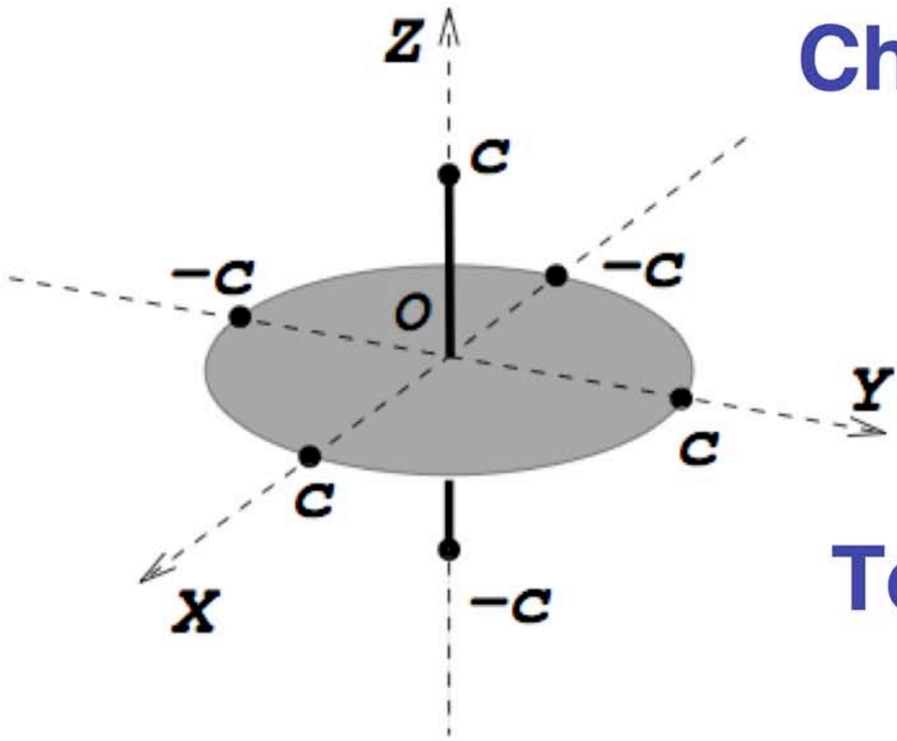
Field lines of mean-field Berry curvature

$$\mathcal{B} = \frac{p^3}{2(cp+Z)^2(cp^3+Z)}(\mathbf{R} + cp\hat{z})$$





Charge of disk-shaped monopole



Charge distribution

$$\rho = \frac{1}{c\sqrt{c^2 - (X^2 + Y^2)}}$$

Total charge

$$Q = \int_{disk} \rho d\tau = \int_S \vec{B} \bullet d\vec{s} = 2\pi$$

Berry phase vs. Hannay's angle

$\hat{H}(X,Y,Z)$	$H(X,Y,Z)$
γ_n	$\Delta\theta$
Berry PRS(1984)	Hannay JPA(1985)

Semiclassical relation

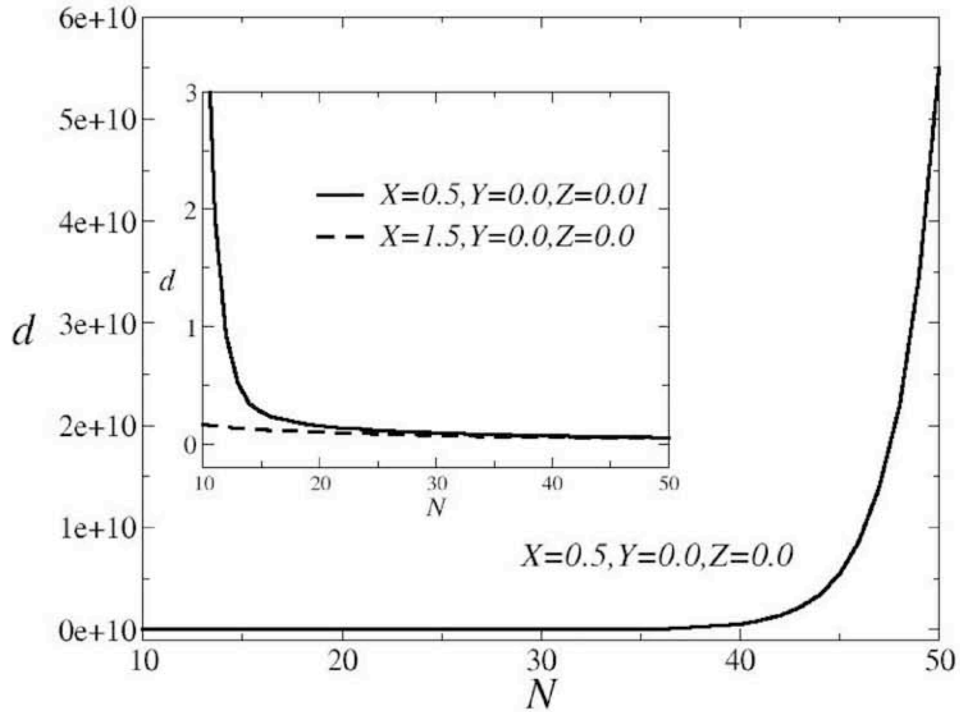
$$\vec{B}_n \quad \overset{\hbar \to 0}{=\!=} \quad \vec{W}$$

Berry JPA(1985)

Breakdown of semiclassical relation

For our system, the semiclassical relation is

$$\lim_{N\to\infty} \delta \mathbf{B} = \lim_{N\to\infty} \left(\frac{\mathrm{B}_N}{N} - \mathcal{B}\right) = 0$$

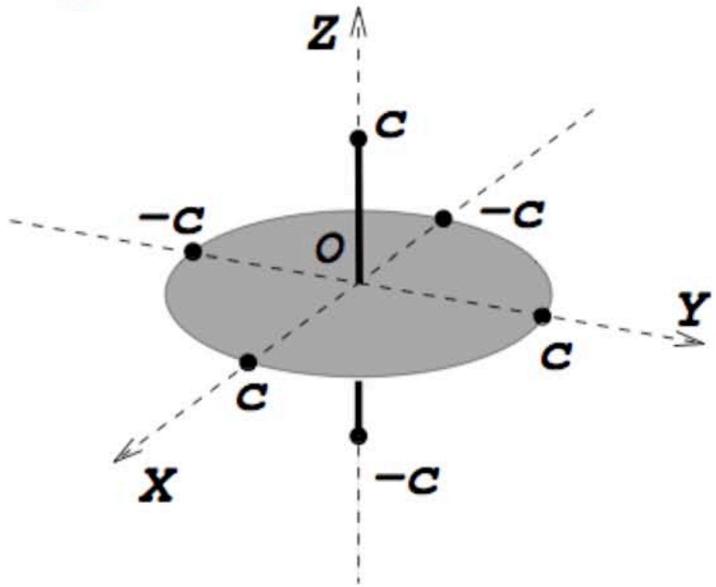


Summary

- Anomalous monopole of disk shape

1. Failure of von Neumann-Wigner theorem in the semiclassical limit

2. Breakdown of the semiclassical relation of Berry curvatures



What are the physical consequences?